

# Estimation of Skill Distributions

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In this paper, we study the problem of learning the skill distribution of a population of agents from observations of pairwise games in a tournament. These games are played among randomly drawn agents from the population. The agents in our model can be individuals, sports teams, or even Wall Street fund managers. Formally, we postulate that the likelihoods of outcomes of games are governed by the parametric Bradley-Terry-Luce (or multinomial logit) model, where the probability of an agent beating another is the ratio between its skill level and the pairwise sum of skill levels, and the skill parameters are drawn from an unknown, non-parametric skill density of interest. The above problem is, in essence, to learn a distribution from noisy and quantized observations. We propose a surprisingly simple and tractable algorithm that learns the skill density with near-optimal minimax mean squared error scaling as  $n^{-1+\epsilon}$ , for any  $\epsilon > 0$ , so long as the density is smooth. Our approach brings together prior work on learning skill parameters from pairwise comparisons with kernel density estimation from non-parametric statistics. Furthermore, we prove information theoretic lower bounds which establish minimax near-optimality of the skill parameter estimation technique used in our algorithm. These bounds utilize a continuum version of Fano’s method along with a careful covering argument. We apply our algorithm to various soccer leagues and world cups, cricket world cups, and mutual funds. We find that the entropy of a learnt distribution provides a quantitative measure of skill, which in turn provides rigorous explanations for popular beliefs about perceived qualities of sporting events, e.g., rankings of soccer leagues. Finally, we apply our method to assess the distribution of skill levels of mutual funds. Our results shed light on the abundance of low quality funds prior to the Great Recession of 2008, and the domination of the industry by more skilled funds after the financial crisis.<sup>†</sup>

*Key words:* Bradley-Terry-Luce model, Parzen-Rosenblatt kernel method, non-parametric density estimation, rank centrality.

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## 1. Introduction

It is a widely-held belief among soccer enthusiasts that the English Premier League (EPL) is the most competitive amongst professional leagues even though the eventual winner is often one of a

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handful of usual suspects (McIntyre 2019, Spacey 2020). Likewise, it is also believed that due to the globalization of the sports industry and increased mobility of talented players, recent Soccer World Cups are far more competitive than earlier ones in the 1970s and '80s. Soccer World Cups are no longer the goal-fests they once were, and the overall standard of play has greatly improved with matches becoming significantly tighter (Scott and Kirk 2018). In a similar vein, the Cricket World Cup in 2019 is considered to be the most exciting in the modern history of the sport, and ended with one of the greatest matches of all time (Smyth et al. 2019, Bull 2019).

But are any of these assertions backed up by data, or are they just common misconceptions? In this work, we answer this question by quantifying such observations, beyond mere sports punditry and subjective opinions, in a data-driven manner. We then illustrate that a similar approach can be used to quantify the evolution of the overall quality and relative skill levels of mutual funds over the years.

To this end, we posit that the population of agents in a tournament, e.g., EPL teams or mutual fund managers, has an associated distribution of skill levels with a probability density function (PDF)  $P_\alpha$  over  $\mathbb{R}_+$ . Our goal is to learn this  $P_\alpha$ . Traditionally, in the non-parametric statistics literature (cf. Tsybakov 2009), one observes samples from the distribution directly to estimate  $P_\alpha$ . In our setting, however, we can only observe extremely noisy, quantized values. Specifically, given  $n \geq 2$  individuals, teams, or players participating in a tournament, indexed by  $[n] \triangleq \{1, \dots, n\}$ , let their latent skill levels be  $\alpha_i$ ,  $i \in [n]$ , which are sampled independently from  $P_\alpha$ . We observe the outcomes of pairwise games or comparisons between them. More precisely, for each  $i \neq j \in [n]$ , independently with probability  $p \in (0, 1]$ , we observe the outcomes of  $k \geq 1$  games, and with probability  $1 - p$ , we observe nothing. Let  $\mathcal{G}(n, p)$  denote the induced Erdős-Rényi random graph with vertices  $[n]$ , where an edge  $\{i, j\} \in \mathcal{G}(n, p)$  (exists in  $\mathcal{G}(n, p)$ ) if and only if games between  $i$  and  $j$  are observed. For  $\{i, j\} \in \mathcal{G}(n, p)$ , let  $Z_m(i, j) \in \{0, 1\}$  denote the outcome of the  $m$ th game between  $i$  and  $j$  for  $m \in [k]$ , with value 1 if  $j$  beats  $i$  and 0 otherwise. By definition,  $Z_m(i, j) + Z_m(j, i) = 1$ . We assume that the likelihoods of the outcomes of games are determined by the *Bradley-Terry-Luce (BTL)* (Bradley and Terry 1952, Luce 1959) or *multinomial logit model* (McFadden 1973) where:

$$\forall i \neq j \in [n], \forall m \in [k], \mathbb{P}(Z_m(i, j) = 1 \mid \alpha_1, \dots, \alpha_n) \triangleq \frac{\alpha_j}{\alpha_i + \alpha_j}, \quad (1)$$

and the  $Z_m(i, j)$ 's (i.e., the outcomes of games) are conditionally independent given  $\alpha_1, \dots, \alpha_n$ . We also note that  $\mathcal{G}(n, p)$  is independent of  $\alpha_1, \dots, \alpha_n$  and the  $Z_m(i, j)$ 's.

Our objective is to learn  $P_\alpha$  from the observations  $\{Z_m(i, j) : \{i, j\} \in \mathcal{G}(n, p), m \in [k]\}$ , instead of  $\alpha_i, i \in [n]$  as in traditional statistics (Tsybakov 2009). For a given, fixed set of  $\alpha_i, i \in [n]$ , learning them from pairwise comparison data  $\{Z_m(i, j) : \{i, j\} \in \mathcal{G}(n, p), m \in [k]\}$  has been extensively studied in the recent literature (Negahban et al. 2012, 2017, Chen et al. 2019). Nevertheless, this line of research does not provide any means to estimate the underlying skill distribution  $P_\alpha$ .

**Table 1** Comparison of our contributions (in blue) with prior works. The notation  $\tilde{O}$  and  $\tilde{\Omega}$  hide  $\text{poly}(\log(n))$  terms, and  $\varepsilon > 0$  is any arbitrarily small constant.

Estimation problem	Loss	Upper bound	Lower bound
Smooth $\mathcal{C}^\infty$ skill PDF	MSE	$\tilde{O}(n^{-1+\varepsilon})$ (Theorem 3)	$\Omega(n^{-1})$ (Tsybakov 2009)
BTL skill parameters	$\ell^\infty$ -norm	$\tilde{O}(n^{-1/2})$ (Chen et al. 2019)	$\tilde{\Omega}(n^{-1/2})$ (Theorem 1)
BTL skill parameters	$\ell^1$ -norm	$O(n^{-1/2})$ (Chen et al. 2019)	$\tilde{\Omega}(n^{-1/2})$ (Theorem 2)

### 1.1. Main Contributions

As the main contribution of this work, we develop a statistically near-optimal and computationally tractable method for estimating the skill distribution  $P_\alpha$  from a subset of pairwise comparisons. Our estimation method is a two-stage algorithm that uses the (spectral) rank centrality estimator (Negahban et al. 2012, 2017) followed by the Parzen-Rosenblatt kernel density estimator (Rosenblatt 1956, Parzen 1962) with carefully chosen bandwidth. We establish that the minimax *mean squared error* (MSE) of our method scales as  $\tilde{O}(n^{-\eta/(\eta+1)})$  for any  $P_\alpha$  belonging to an  $\eta$ -Hölder class in Theorem 3. Thus, if  $P_\alpha$  is smooth ( $\mathcal{C}^\infty$ ) with bounded derivatives, then the minimax MSE is  $\tilde{O}(n^{-1+\varepsilon})$  for any  $\varepsilon > 0$ ; see the discussion after Theorem 3 in Section 3.2. Somewhat surprisingly, although we do not directly observe  $\alpha_i, i \in [n]$ , this minimax MSE rate matches the minimax MSE lower bound of  $\Omega(n^{-1})$  for smooth  $P_\alpha$  even when  $\alpha_i, i \in [n]$  are observed (Ibragimov and Khas'minskii 1982, Tsybakov 2009).

As a key step in our estimation method, we utilize the rank centrality algorithm (Negahban et al. 2012, 2017) for estimating  $\alpha_i, i \in [n]$ . While the optimal learning rate of the rank centrality algorithm with respect to relative  $\ell^2$ -loss is well-understood (Negahban et al. 2012, 2017, Chen et al. 2019), the optimal learning rates with respect to relative  $\ell^\infty$  and  $\ell^1$ -losses are not known since we only know upper bounds (Chen et al. 2019), but not matching minimax lower bounds. In Theorems 1 and 2, we prove minimax lower bounds of  $\tilde{\Omega}(n^{-1/2})$  with respect to both relative  $\ell^\infty$  and  $\ell^1$ -losses. These bounds match the learning rates of the rank centrality algorithm obtained in (Chen et al. 2019) with respect to both  $\ell^\infty$  and  $\ell^1$ -losses, and hence, identify the optimal minimax rates (up to logarithmic factors). We derive these information theoretic lower bounds by employing a recent variant of the generalized Fano's method with covering arguments. Our main technical results are all delineated in Table 1.

Finally, we illustrate the utility of our algorithm through four experiments on real-world data: Cricket World Cups, Soccer World Cups, European soccer leagues, and US mutual funds. Intuitively, a concentrated skill distribution, i.e., one that is close to a Dirac delta measure, corresponds to a balanced tournament with players that are all roughly equally skilled. Hence, the outcomes of games are quite random or unpredictable. On the other hand, a skill distribution that is close to

uniform suggests a wider spread of players' skill levels. So, the outcomes of games are driven more by skill rather than luck (or random chance). We, therefore, propose to use the negative (differential) entropy of a learnt skill distribution as a way to measure the "overall skill score," because entropy captures the holistic level of spread or variation of a PDF. For Cricket World Cups, we find that negative entropy decreases from 2003 to 2019. Indeed, this corroborates with fan experience, where in 2003, Australia and India dominated but all other teams were roughly equal, while in 2019, there was a healthy spread of skill levels making many teams potential contenders for the championship. In soccer, we observe that the EPL and World Cup have high negative entropy, which indicates that most teams are competitive, and thus, it is very difficult to predict outcomes up front. Furthermore, we see that the negative entropy of World Cups has increased over the years, giving credence to the belief that World Cups have gotten more competitive over time. Lastly, the negative entropy of US mutual funds decreases significantly during the Great Recession of 2008, and we see flatter skill distributions post 2008. This reveals that mutual funds became more competent to avoid being weeded out of the market by the financial crisis.

It is worth mentioning that there are several reasons to estimate  $P_\alpha$  rather than the individual skill levels  $\alpha_1, \dots, \alpha_n$ . Although specific functionals of  $P_\alpha$ , e.g., moments or variance, may be directly estimated from estimates of skill levels, estimating  $P_\alpha$  simultaneously recovers information about all such functionals. Indeed, it is shown in Proposition 4 that MSE guarantees for estimating  $P_\alpha$  yield uniform guarantees on estimating bounded statistics of the form  $\mathbb{E}[f(\alpha)]$  for functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ . Since different functionals are pertinent for different applications, a good estimate of the skill distribution  $P_\alpha$  is very useful. For example, we utilize negative entropy of  $P_\alpha$  to define overall skill scores. Standard non-parametric plug-in estimators for entropy in the literature, e.g., integral, resubstitution, or splitting data estimators (Beirlant et al. 1997), require an estimate of  $P_\alpha$  to compute entropy. Therefore, in the context of this work, estimating  $P_\alpha$  is eminently desirable.

## 1.2. Related Literature

The problem of estimating distributions of skill levels from tournaments has received increased attention due to the recent advent of fantasy sports platforms, which give rise to new legal and policy making challenges concerned with regulating the accompanying rise of gambling on such platforms (cf. Getty et al. 2018, and follow-up work). Indeed, when the distribution of skill levels of players is concentrated around one point, the associated game is essentially one of chance (or luck), and governments may understandably seek to place more betting regulations on such tournaments. While Getty et al. (2018) provide an empirical study of an ad hoc measure of skill using fantasy sports data, we consider a rigorous statistical formulation of this problem where the objective is to estimate an unknown PDF of skill levels from partially observed win-loss data of tournaments.

As mentioned earlier, we assume that all players in a tournament have latent “skill” or “merit” parameters that are drawn from an unknown prior skill PDF, and these skill parameters determine the likelihoods of wins and losses in games according to the BTL model. Our algorithm to estimate such skill distributions proceeds by first estimating skill parameters from the observed data, and then estimating the skill distribution based on these parameter estimates. To estimate the skill PDFs from (estimated) skill parameters in the second stage of our algorithm, we exploit kernel density estimation techniques that were originally developed in (Rosenblatt 1956, Parzen 1962, Epanechnikov 1969). Moreover, as noted in Table 1, to evaluate the minimax MSE risk achieved by our algorithm, we compare our MSE risk scaling with well-known minimax lower bounds on density estimation for certain classes of analytic densities (cf. Ibragimov and Khas’minskii 1982, and the references therein). On a separate front, to establish the near-optimality of the skill parameter estimation technique (to be explained in due course) used in our algorithm, we exploit a variant of the generalized Fano’s method. This method was also initially developed in the context of density estimation in (Ibragimov and Khas’minskii 1977, Khas’minskii 1979). For the sake of brevity, we do not review the extensive non-parametric density estimation literature any further, and instead refer readers to (Tsybakov 2009, Chapters 1 and 2), (Wasserman 2019), and the references therein for thorough modern treatments.

Since we assume that the likelihoods of the outcomes of two-player games in a tournament follow the BTL model (Bradley and Terry 1952, Luce 1959), and estimation of the skill parameters of this model forms the first stage of our proposed algorithm, we outline several relevant aspects of the vast literature concerning the BTL model in the remainder of this section. Indeed, while the BTL model was introduced in statistics to study pairwise comparisons (Bradley and Terry 1952), it has a long and diverse history. The model was initially proposed by Zermelo (1929) in the context of chess, and he also provided an iterative algorithm to compute the maximum likelihood (ML) estimators of the BTL skill parameters. Moreover, the BTL model is a special case of the *Plackett-Luce (PL) model* (Luce 1959, Plackett 1975), which was originally developed in mathematical psychology. The PL model defines a probability distribution over rankings (or permutations) of players that is a natural consequence of *Luce’s choice axiom*. This axiom can be perceived as a formulation of the “*independence of irrelevant alternatives*” in social choice theory and econometrics. In fact, the work of McFadden (1973) on the multinomial logit model in economics is equivalent to the PL model. The earliest known model that is related to the PL model is perhaps the (generalized parametric) *Thurstonian model* from psychometrics, which provides a probability distribution over rankings using the so called *law of comparative judgment* (Thurstone 1927). Specifically, Thurstone models a “discriminal process” to rank  $n$  items by first associating latent merit parameters  $\omega_1, \dots, \omega_n$  to each of the  $n$  items, and then ranking them by ranking the corresponding random variables

$\omega_1 + X_1, \dots, \omega_n + X_n$ , where the independent and identically distributed (i.i.d.) random variables  $X_1, \dots, X_n$  represent “noise” in the discriminial process. As explained in (Diaconis 1988, Section 9D), the resulting distribution over rankings is equivalent to the PL model when the  $X_i$ ’s have Gumbel (or generalized extreme value type-I) distribution (cf. Yellott, Jr. 1977). We refer readers to (Diaconis 1988, Sections 9C and 9D) for other models of rankings based on *exponential families* and further equivalent formulations of the BTL and PL models, and to (Shah 2019) for a comprehensive discussion on other equivalent models from a modern machine learning perspective. For example, the celebrated *Boltzmann-Gibbs distribution* in statistical physics and the *softmax* model in machine learning are also versions of the PL model.

In order to estimate the skill parameters of the BTL model, two main families of algorithms have been developed in the literature. The first of these is a class of *minorization-maximization (MM) algorithms* that generalize Zermelo’s iterative algorithm in (Zermelo 1929). Much like how Zermelo’s algorithm computes ML estimators of the parameters under a *strong connectivity condition* (Ford, Jr. 1957) (also see (Hunter 2004, Assumption 1) for a graph theoretic interpretation), the more general MM algorithms can be utilized to perform ML estimation for “generalized” BTL models (Hunter 2004). Moreover, although MM algorithms are typically seen as extending the better known *expectation-maximization (EM) algorithms* for ML estimation of latent variable models, e.g., Gaussian mixture models (Dempster et al. 1977), the MM algorithms for generalized BTL models can also be construed as special cases of EM algorithms corresponding to certain choices of latent variables (Caron and Doucet 2012). In contrast, in this paper, we utilize the second, more recently discovered, family of spectral algorithms based on the notion of rank centrality introduced in (Negahban et al. 2012, 2017). The main innovation of such spectral algorithms is to construe (normalized) skill parameters as an invariant distribution of a reversible Markov chain, and armed with this perspective, estimate skill parameters by first estimating the stochastic kernel defining the Markov chain.

Both MM and spectral algorithms have been analyzed extensively in the literature. For instance, Simons and Yao (1999) prove the consistency and asymptotic normality of ML estimators for skill parameters computed by Zermelo’s algorithm, and Negahban et al. (2017, Theorems 1 and 2) establish sample complexity bounds for the relative  $\ell^2$ -norm estimation error of (normalized) skill parameters. Furthermore, both families of algorithms are shown to be optimal for recovering the top few ranked players in (Chen et al. 2019), which presents non-asymptotic analysis for relative  $\ell^\infty$  and  $\ell^2$ -norm losses. In particular, Negahban et al. (2017) and Chen et al. (2019) assume that a random Erdős-Rényi graph captures the subset of pairwise games that are observed in a tournament. Our analysis also considers this partial observation model, and exploits the relative  $\ell^\infty$  and  $\ell^2$ -norm loss results of (Chen et al. 2019). In a different vein, Shah et al. (2016) establish

minimax estimation bounds for squared semi-norm losses defined by graph Laplacian matrices, where the fixed graphs encode the subsets of observed pairwise games (also see follow-up work), Chatterjee (2015) demonstrates that the *universal singular value thresholding* algorithm can be used to estimate (stochastically transitive) “non-parametric” BTL models, and Hendrickx et al. (2020) analyze a *weighted least squares* method to estimate BTL skill parameters under a chordal (or projection) distance loss. Finally, we refer readers to (Guiver and Snelson 2009, Caron and Doucet 2012) and the references therein for other recent research on efficient Bayesian inference for BTL and PL models. As opposed to these works, we analyze minimax estimation of BTL models under a previously unexplored setting where skill parameters are drawn i.i.d. from a prior skill PDF.

### 1.3. Notational Preliminaries

We briefly introduce some relevant notation. Let  $\mathbb{N} \triangleq \{1, 2, 3, \dots\}$  denote the set of natural numbers. For any  $n \in \mathbb{N}$ , let  $\mathcal{S}_n$  denote the probability simplex of row probability vectors in  $\mathbb{R}^n$ , and  $\mathcal{S}_{n \times n}$  denote the set of all  $n \times n$  row stochastic matrices in  $\mathbb{R}^{n \times n}$ . For any vector  $x \in \mathbb{R}^n$  and any  $q \in [1, \infty]$ , let  $\|x\|_q$  denote the  $\ell^q$ -norm of  $x$ , and for any matrix  $A \in \mathbb{R}^{n \times n}$ , let  $\det(A)$  denote the determinant of  $A$ . Moreover,  $\exp(\cdot)$  and  $\log(\cdot)$  denote the natural exponential and logarithm functions with base  $e$ , respectively,  $\mathbb{1}\{\cdot\}$  denotes the indicator function that equals 1 if its input proposition is true and 0 otherwise, and  $\lceil \cdot \rceil$  and  $\lfloor \cdot \rfloor$  denote the ceiling and floor functions, respectively. Finally, we will use standard Bachmann-Landau asymptotic notation, e.g.,  $O(\cdot)$ ,  $\Omega(\cdot)$ ,  $\Theta(\cdot)$ , where it is understood that  $n \rightarrow \infty$ , and tilde notation, e.g.,  $\tilde{O}(\cdot)$ ,  $\tilde{\Omega}(\cdot)$ ,  $\tilde{\Theta}(\cdot)$ , when we neglect  $\text{poly}(\log(n))$  factors and problem parameters other than  $n$ .

Our ensuing analysis will utilize the joint probability law  $\mathbb{P}(\cdot)$  of  $\alpha_1, \dots, \alpha_n$ ,  $\mathcal{G}(n, p)$ , and  $\{Z_m(i, j) : \{i, j\} \in \mathcal{G}(n, p), m \in [k]\}$  described earlier, and its associated expectation operator  $\mathbb{E}[\cdot]$  (both of which depend on the unknown PDF  $P_\alpha$ ). Moreover, unless stated otherwise, our results will hold for all relevant PDFs  $P_\alpha$  that define  $\mathbb{P}$ .

### 1.4. Organization

In closing Section 1, we delineate how the remainder of the paper is organized. In Section 2, we further clarify the formal setup and describe our estimation algorithm that learns  $P_\alpha$  from partial observations of games. In Section 3, we present our main results, i.e., Theorems 1, 2, and 3 (mentioned earlier), as well as several auxiliary results. Then, we derive our MSE upper bound on skill PDF estimation (Theorem 3) and associated auxiliary results in Section 4, and we prove our minimax skill parameter estimation results (Theorems 1 and 2) and associated auxiliary results in Section 5. Finally, we illustrate several simulation results pertaining to Cricket World Cups, Soccer

World Cups, European soccer leagues, and US mutual funds in Section 6, and then conclude our discussion in Section 7.

We remark that (as indicated at the outset of this paper) the statements of the technical results in Section 3, except Proposition 4, and the numerical simulation results in Section 6, except those in Section 6.4 (which have been significantly extended), have appeared in the conference proceedings (Jadbabaie et al. 2020). However, all other technical results, simulations, and all proofs (e.g., in Sections 4 and 5) are the contributions of this paper.

## 2. Estimation Algorithm

Our interest is in estimating the skill PDF  $P_\alpha$  from noisy, discrete observations  $\{Z_m(i, j) : \{i, j\} \in \mathcal{G}(n, p), m \in [k]\}$ . Instead, if we had exact knowledge of the samples  $\alpha_i, i \in [n]$  from  $P_\alpha$ , then we could utilize traditional methods from non-parametric statistics such as kernel density estimation. However, we do not have access to these samples. So, given pairwise comparisons  $\{Z_m(i, j) : \{i, j\} \in \mathcal{G}(n, p), m \in [k]\}$  generated as per the BTL model with parameters  $\alpha_i, i \in [n]$ , we can use some recent developments from the BTL-related literature to estimate these skill parameters first. Therefore, a natural two-stage algorithm is to first estimate  $\alpha_i, i \in [n]$  using the observations, and then use these estimated parameters to produce an estimate of  $P_\alpha$ . We do precisely this. The key challenge is to ensure that the PDF estimation method is robust to the estimation error in  $\alpha_i, i \in [n]$ . As our main contribution, we rigorously argue that carefully chosen methods for both steps produces as good an estimation of  $P_\alpha$  as if we had access to the exact knowledge of  $\alpha_i, i \in [n]$ .

### 2.1. Formal Setup

We formalize the setup here. For any given  $\delta, \epsilon, b \in (0, 1)$  and  $\eta, L_1, B > 0$ , let  $\mathcal{P} = \mathcal{P}(\delta, \epsilon, b, \eta, L_1, B)$  be the (non-parametric) set of all uniformly bounded PDFs with respect to the Lebesgue measure on  $\mathbb{R}$  that have support in  $[\delta, 1]$ , belong to the  $\eta$ -Hölder class (Tsybakov 2009, Definition 1.2), and are lower bounded by  $b$  in an  $\epsilon$ -neighborhood of 1. More precisely, every PDF  $f \in \mathcal{P}$  with support in  $[\delta, 1]$  satisfies:

1.  $f$  is bounded (almost everywhere), i.e.,  $f(x) \leq B$  for all  $x \in [\delta, 1]$ ,
2.  $f$  is  $s = \lceil \eta \rceil - 1$  times differentiable, and its  $s$ th derivative  $f^{(s)} : [\delta, 1] \rightarrow \mathbb{R}$  is  $(\eta - s)$ -Hölder continuous:

$$\forall x, y \in [\delta, 1], \quad |f^{(s)}(x) - f^{(s)}(y)| \leq L_1 |x - y|^{\eta - s}, \quad (2)$$

3.  $f(x) \geq b$  for all  $x \in [1 - \epsilon, 1]$ .



As an example, when  $\eta = 1$ ,  $\mathcal{P}$  denotes the set of all  $L_1$ -Lipschitz continuous PDFs on  $[\delta, 1]$  that are lower bounded in the neighborhood of unity. Furthermore, we define the observation matrix  $Z \in [0, 1]^{n \times n}$ , whose  $(i, j)$ th entry is:

$$\forall i, j \in [n], \quad Z(i, j) \triangleq \begin{cases} \mathbb{1}\{\{i, j\} \in \mathcal{G}(n, p)\} \frac{1}{k} \sum_{m=1}^k Z_m(i, j), & i \neq j, \\ 0, & i = j. \end{cases} \quad (3)$$

**2.1.1. Minimax Estimation Error Formulation.** It turns out that  $Z$  is a sufficient statistic for the purposes of estimating  $\alpha_i$ ,  $i \in n$  (Chen et al. 2019, p.2208). For this reason, we shall restrict our attention to all possible estimators of  $P_\alpha$  using  $Z$ . Specifically, let  $\hat{\mathcal{P}}$  be set of all possible measurable and potentially randomized estimators that map  $Z$  to a Borel measurable function from  $\mathbb{R}$  to  $\mathbb{R}$ . Then, the minimax MSE risk is defined as:

$$R_{\text{MSE}}(n) \triangleq \inf_{\hat{P} \in \hat{\mathcal{P}}} \sup_{P_\alpha \in \mathcal{P}} \mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{P}(x) - P_\alpha(x) \right)^2 dx \right] \quad (4)$$

where the expectation is with respect to the randomness in  $Z$  as well as within the estimator. Our interest will be in understanding the scaling of  $R_{\text{MSE}}(n)$  as a function of  $n$  and  $\eta$ . In the sequel, we will assume that the parameters  $k, p, \delta, \epsilon, b$  can depend on  $n$ , and all other parameters are constant.

**2.1.2. Why These Assumptions?** We also briefly explain the motivations behind the assumptions we impose on the non-parametric class of skill densities  $\mathcal{P}$ . By restricting  $P_\alpha$  to  $\mathcal{P}$ , we are able to perform tractable non-asymptotic analysis in the sequel. Indeed, the Hölder class and boundedness assumptions of  $\mathcal{P}$  are standard in the non-parametric density estimation literature (see, e.g., Tsybakov 2009, Section 1.2). Moreover, since the BTL likelihoods in (1) are invariant to scaling the skill parameters, we may assume without loss of generality that  $\alpha_1, \dots, \alpha_n \leq 1$ . (The lower bound assumption in a neighborhood of unity ensures that at least one  $\alpha_i$  is close to the upper bound of unity with high probability.) On the other hand, assuming that  $\alpha_1, \dots, \alpha_n \geq \delta$  is equivalent to the *condition number* (or dynamic range) bound:

$$\max_{i, j \in [n]} \frac{\alpha_i}{\alpha_j} \leq \frac{1}{\delta}, \quad (5)$$

which is very often exploited in the BTL-related literature; see, e.g., (Simons and Yao 1999, Equations (1.5) and (1.6)), (Negahban et al. 2017, Theorems 1 and 2), or (Chen et al. 2019, Equation (2.4)). Note that  $P_\alpha$  having support in  $[\delta, 1]$  corresponds precisely to the condition that  $\alpha_1, \dots, \alpha_n \in [\delta, 1]$ .

## 2.2. Description of Algorithm

We now propose an algorithm that constructs a good estimator  $\hat{P}^*$  of the PDF  $P_\alpha$  based on  $Z$ . By analyzing this algorithm, we will eventually obtain an upper bound on (4).

**2.2.1. Step 1: Estimate  $\alpha_i, i \in [n]$ .** Given the observation matrix  $Z$ , let  $S \in \mathbb{R}^{n \times n}$  be the “empirical stochastic matrix” whose  $(i, j)$ th element is given by:

$$\forall i, j \in [n], S(i, j) \triangleq \begin{cases} \frac{1}{2np} Z(i, j), & i \neq j, \\ 1 - \frac{1}{2np} \sum_{r=1}^n Z(i, r), & i = j. \end{cases} \quad (6)$$

The ensuing proposition shows that  $S$  is indeed row stochastic with high probability when  $p = \Omega(\log(n)/n)$ .

**PROPOSITION 1 (Empirical Stochastic Matrix).** *If  $n \geq 2$  and  $p \geq 16(c_1 + 1) \log(n)/(3n)$  for any fixed (universal) constant  $c_1 > 0$ , then we have:*

$$\mathbb{P}(S \in \mathcal{S}_{n \times n}) \geq 1 - \frac{1}{n^{c_1}}.$$

Proposition 1 is proved in Appendix A.1. Next, inspired by the *rank centrality* algorithm in (Negahban et al. 2012, 2017), let  $\hat{\pi}_* \in \mathcal{S}_n$  be the invariant distribution of  $S$ , given by:

$$\hat{\pi}_* \triangleq \begin{cases} \text{invariant distribution of } S \text{ such that } \hat{\pi}_* = \hat{\pi}_* S, & S \in \mathcal{S}_{n \times n} \\ \text{any randomly chosen distribution in } \mathcal{S}_n, & S \notin \mathcal{S}_{n \times n} \end{cases} \quad (7)$$

where when  $S \in \mathcal{S}_{n \times n}$ , an invariant distribution always exists and we choose one arbitrarily when it is not unique. Then, we can define the following estimates of  $\alpha_1, \dots, \alpha_n$  based on  $Z$ :

$$\forall i \in [n], \hat{\alpha}_i \triangleq \frac{\hat{\pi}_*(i)}{\|\hat{\pi}_*\|_\infty} \quad (8)$$

where  $\hat{\pi}_*(i)$  denotes the  $i$ th entry of  $\hat{\pi}_*$  for  $i \in [n]$ .

**2.2.2. Step 2: Estimate  $P_\alpha$ .** Using (8), we construct the *Parzen-Rosenblatt (PR) kernel density estimator*  $\hat{\mathcal{P}}^* : \mathbb{R} \rightarrow \mathbb{R}$  for  $P_\alpha$  based on  $\hat{\alpha}_1, \dots, \hat{\alpha}_n$  instead of  $\alpha_1, \dots, \alpha_n$  (Rosenblatt 1956, Parzen 1962):

$$\forall x \in \mathbb{R}, \hat{\mathcal{P}}^*(x) \triangleq \frac{1}{nh} \sum_{i=1}^n K\left(\frac{\hat{\alpha}_i - x}{h}\right) \quad (9)$$

where  $h > 0$  is a judiciously chosen bandwidth parameter (see the proof in Section 4.2):

$$h = \gamma \max \left\{ \frac{1}{\delta^{\frac{1}{\eta+1}} (pk)^{\frac{1}{2\eta+2}}}, 1 \right\} \left( \frac{\log(n)}{n} \right)^{\frac{1}{2\eta+2}} \quad (10)$$

for any (universal) constant  $\gamma > 0$ , and  $K : [-1, 1] \rightarrow \mathbb{R}$  is any fixed Lipschitz continuous kernel function of order  $s = \lceil \eta \rceil - 1$ , which we define below.

For any  $s \in \mathbb{N} \cup \{0\}$ , the function  $K : [-1, 1] \rightarrow \mathbb{R}$  is said to be a *kernel of order  $s$*  if it satisfies the following conditions:

1.  $K(x) = 0$  for  $|x| > 1$ ,

2.  $K$  is (Lebesgue) square-integrable, i.e.,  $\int_{\mathbb{R}} K(x)^2 dx < \infty$ ,
3.  $\int_{\mathbb{R}} K(x) dx = 1$ ,
4.  $\int_{\mathbb{R}} x^i K(x) dx = 0$  for all  $i \in [s]$  when  $s \geq 1$ .

Note that  $K$  is integrable since it is square-integrable and has support in  $[-1, 1]$ , and any map  $\mathbb{R} \ni x \mapsto x^r K(x)$  with  $r \geq 0$  is integrable because  $K$  is integrable and has support in  $[-1, 1]$ . Such kernels of order  $s$  can be constructed using orthogonal polynomials as expounded in (Tsybakov 2009, Section 1.2.2). We will additionally assume that:

5. There exists a constant  $L_2 > 0$  such that our kernel  $K : [-1, 1] \rightarrow \mathbb{R}$  is  $L_2$ -Lipschitz continuous:

$$\forall x, y \in \mathbb{R}, \quad |K(x) - K(y)| \leq L_2 |x - y|. \quad (11)$$

This is a mild assumption since several well-known kernels satisfy it. For instance, the (parabolic) Epanechnikov kernel (Epanechnikov 1969):

$$\forall x \in \mathbb{R}, \quad K_E(x) \triangleq \frac{3}{4} (1 - x^2) \mathbb{1}\{|x| \leq 1\} \quad (12)$$

has order  $s = 1$ , and is Lipschitz continuous with  $L_2 = \frac{3}{2}$ . Other examples of valid kernels can be found in (Tsybakov 2009, p.3 and Section 1.2.2).

**2.2.3. Summary of Algorithm.** Here, we provide the pseudo-code summary of our algorithm.

---

**Algorithm 1** Estimating skill PDF  $P_\alpha$  using  $Z$ .

---

**Input:** Observation matrix  $Z \in [0, 1]^{n \times n}$  (as defined in (3))

**Output:** Estimator  $\hat{\mathcal{P}}^* : \mathbb{R} \rightarrow \mathbb{R}$  of the unknown PDF  $P_\alpha$

**Step 1: Skill parameter estimation using rank centrality algorithm**

- 1: Construct  $S \in \mathcal{S}_{n \times n}$  according to (6) using  $Z$  (and  $p$  and  $n$ )
- 2: Compute leading left eigenvector  $\hat{\pi}_* \in \mathcal{S}_n$  of  $S$  in (7)  $\triangleright \hat{\pi}_*$  is the invariant distribution of  $S$
- 3: Compute estimates  $\hat{\alpha}_i = \hat{\pi}_*(i) / \|\hat{\pi}_*\|_\infty$  for  $i = 1, \dots, n$  via (8)

**Step 2: Kernel density estimation using Parzen-Rosenblatt method**

- 4: Compute bandwidth  $h$  via (10) (using  $p$ ,  $k$ ,  $\delta$ ,  $\eta$ , and  $n$ )
  - 5: Construct  $\hat{\mathcal{P}}^*$  according to (9) using  $\hat{\alpha}_1, \dots, \hat{\alpha}_n$ ,  $h$ , and a valid kernel  $K : [-1, 1] \rightarrow \mathbb{R}$
  - 6: **return**  $\hat{\mathcal{P}}^*$
- 

With fixed  $\delta \in (0, 1)$ ,  $\eta > 0$ , and a valid kernel  $K : [-1, 1] \rightarrow \mathbb{R}$ , and given knowledge of  $k \in \mathbb{N}$  and  $p \in (0, 1]$  (which can also be easily estimated), Algorithm 1 constructs the estimator (9) for  $P_\alpha$  based on  $Z$ . In Algorithm 1, we assume that  $S \in \mathcal{S}_{n \times n}$ , because this is almost always the case in practice. Furthermore, if  $k$  varies between players so that  $i$  and  $j$  play  $k_{i,j} = k_{j,i}$  games for  $i \neq j \in [n]$ , we

can re-define the data  $Z(i, j)$  to use  $k_{i,j}$  instead of  $k$  in (3), and utilize an appropriately altered bandwidth  $h$ . For example, we can use  $k' = \min_{\{i,j\} \in \mathcal{G}(n,p)} k_{i,j}$  in place of  $k$  in (10) to define  $h$ , which would yield theoretical guarantees akin to Theorem 3 with  $k'$ . The computational complexity of Algorithm 1 is determined by the running time of rank centrality, e.g., if the spectral gap of  $S$  is  $\Theta(1)$  and we use *power iteration* (cf. Golub and van Loan 1996, Demmel 1997, Sections 7.3.1 and 4.4.1, respectively) to obtain an  $O(\text{poly}(n^{-1}))$   $\ell^2$ -approximation of  $\hat{\pi}_*$ , then Algorithm 1 runs in  $O(n^2 \log(n))$  time.

### 2.3. Intuition for Algorithm

We briefly explain the intuition behind each of the two steps of Algorithm 1. As we mentioned earlier, Step 1 of Algorithm 1 is inspired by the (spectral) rank centrality algorithm of Negahban et al. (2012, 2017). To understand this stage, define the row stochastic matrix  $D \in \mathcal{S}_{n \times n}$ , whose  $(i, j)$ th element is given by the BTL skill parameters:

$$\forall i, j \in [n], \quad D(i, j) \triangleq \begin{cases} \frac{1}{2n} \left( \frac{\alpha_j}{\alpha_i + \alpha_j} \right), & i \neq j \\ 1 - \frac{1}{2n} \sum_{r \in [n] \setminus \{i\}} \frac{\alpha_r}{\alpha_i + \alpha_r}, & i = j \end{cases} \quad (13)$$

where  $D(i, j) + D(j, i) = (2n)^{-1}$  for all  $i, j \in [n]$  such that  $i \neq j$ . Note that it is straightforward to verify from (13) that  $D(i, i) \geq 0$  for all  $i \in [n]$ , because:

$$\begin{aligned} \forall i \in [n], \quad D(i, i) &= 1 - \frac{1}{2n} \sum_{r \in [n] \setminus \{i\}} \frac{\alpha_r}{\alpha_i + \alpha_r} \\ &= \frac{1}{2n} \sum_{r \in [n] \setminus \{i\}} \frac{2n}{n-1} - \frac{\alpha_r}{\alpha_i + \alpha_r} \\ &= \frac{1}{2n} \sum_{r \in [n] \setminus \{i\}} \frac{n+1}{n-1} + \frac{\alpha_i}{\alpha_i + \alpha_r} \\ &= \frac{n+1}{2n} + \frac{1}{2n} \sum_{r \in [n] \setminus \{i\}} \frac{\alpha_i}{\alpha_i + \alpha_r} \geq 0. \end{aligned} \quad (14)$$

We will construe  $D$  as the transition probability matrix of a (time-homogeneous) discrete-time Markov chain on the state space  $[n]$  of players. Furthermore, define the ‘‘canonically scaled’’ skill parameters  $\pi \in \mathcal{S}_n$  with  $i$ th entry given by:

$$\forall i \in [n], \quad \pi(i) \triangleq \frac{\alpha_i}{\alpha_1 + \dots + \alpha_n}. \quad (15)$$

Clearly,  $\pi$  can also be used to define the BTL likelihoods in (1) instead of  $\alpha_1, \dots, \alpha_n$ , since  $\alpha_j / (\alpha_i + \alpha_j) = \pi(j) / (\pi(i) + \pi(j))$  for all  $i \neq j \in [n]$ . Next, following the crucial observation of

Negahban et al. (2017), notice using (13) and (15) that the ensuing *detailed balance conditions* are satisfied:

$$\forall i, j \in [n], \quad \pi(i)D(i, j) = \pi(j)D(j, i). \quad (16)$$

This implies that  $D$  defines a *reversible* Markov chain with invariant distribution  $\pi = \pi D$  (see, e.g., Levin et al. 2009, Proposition 1.19). Moreover, this Markov chain is ergodic (i.e., irreducible and aperiodic) because  $D > 0$  entry-wise, which means that  $\pi$  is the unique invariant distribution of  $D$ . This general idea that canonically scaled skill parameters of the BTL model form an invariant distribution of a reversible Markov chain is known as “rank centrality” (Negahban et al. 2012, 2017).

Step 1 of Algorithm 1 estimates the BTL skill parameters  $\alpha_1, \dots, \alpha_n$  using  $\hat{\alpha}_1, \dots, \hat{\alpha}_n$  in (8) by first estimating the canonically scaled skill parameters  $\pi \in \mathcal{S}_n$  in (15). To estimate  $\pi$ , it is reasonable to first produce an estimate of  $D$  that is itself a row stochastic matrix (with high probability), and then utilize the corresponding invariant distribution as our estimate of  $\pi$ . Notice that:

$$\mathbb{E}[S | \alpha_1, \dots, \alpha_n] = D \quad (17)$$

where  $S \in \mathbb{R}^{n \times n}$  is defined in (6). Hence,  $S$  (which is row stochastic with high probability) can be construed as our estimator of the Markov kernel  $D$ . As a consequence, the invariant distribution  $\hat{\pi}_* \in \mathcal{S}_n$  of  $S$  in (7) can be perceived as our estimator of  $\pi$ .

Step 2 of Algorithm 1 constructs an estimator for the unknown PDF  $P_\alpha$  of interest using the skill parameter estimates  $\hat{\alpha}_1, \dots, \hat{\alpha}_n$  obtained from Step 1. Clearly, if we had access to the true i.i.d. samples  $\alpha_1, \dots, \alpha_n$  from  $P_\alpha$ , then we could use the vanilla PR kernel density estimator (see, e.g., (19) in Section 3.2) to estimate  $P_\alpha$ , because it is known to be minimax optimal for appropriate choices of bandwidth  $h$  and kernel function  $K$  (cf. Tsybakov 2009, Wasserman 2019). However, we do not have access to these true samples. Thus, we utilize the estimates  $\hat{\alpha}_1, \dots, \hat{\alpha}_n$  to construct an analogous estimator in (9), which is the output of (Step 2 of) Algorithm 1. Intuitively, we expect this estimator to perform well, because  $\hat{\alpha}_1, \dots, \hat{\alpha}_n$  should be “close” to  $\alpha_1, \dots, \alpha_n$  when  $n$  is large.

### 3. Main Results and Discussion

We now present our main results: an achievable minimax MSE for the  $P_\alpha$  estimation method in Algorithm 1, and minimax lower bounds on estimation of the skill parameters  $\alpha_i, i \in [n]$  from  $Z$  (i.e., Step 1 of Algorithm 1) for *any* method. This collectively establishes the near-optimality of our proposed method as  $\eta \rightarrow \infty$ , i.e., as the density becomes smooth ( $\mathcal{C}^\infty$ ). To this end, we first establish minimax rates for skill parameter estimation, and then derive minimax rates for PDF estimation.

### 3.1. Tight Minimax Bounds on Skill Parameter Estimation

To obtain tight  $P_\alpha$  estimation, it is essential that we have tight skill parameter estimation. Hence, we show that the parameter estimation step performed in (7) has minimax optimal rate. Specifically, we consider the canonically scaled skill parameters  $\pi \in \mathcal{S}_n$  given by (15), which equivalently define the BTL model (1). Building upon (Chen et al. 2019, Theorem 3.1), the ensuing theorem portrays that the *minimax relative  $\ell^\infty$ -risk* of estimating  $\pi$  based on  $Z$  is  $\tilde{\Theta}(n^{-1/2})$  (see Table 1). For simplicity, we will assume throughout Section 3.1 on skill parameter estimation that  $\delta$ ,  $p$ , and  $k$  are  $\Theta(1)$ .

**THEOREM 1 (Minimax Relative  $\ell^\infty$ -Risk).** *For sufficiently large constants  $c_{14}, c_{15} > 0$  (which depend on  $\delta$ ,  $p$ , and  $k$ ), and for all sufficiently large  $n \in \mathbb{N}$ :*

$$\frac{c_{14}}{\log(n)\sqrt{n}} \leq \inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}} \mathbb{E} \left[ \frac{\|\hat{\pi} - \pi\|_\infty}{\|\pi\|_\infty} \right] \leq \sup_{P_\alpha \in \mathcal{P}} \mathbb{E} \left[ \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \right] \leq c_{15} \sqrt{\frac{\log(n)}{n}}$$

where the infimum is over all estimators  $\hat{\pi} \in \mathcal{S}_n$  of  $\pi$  based on  $Z$ , and  $\hat{\pi}_* \in \mathcal{S}_n$  is defined in (7).

The proof of Theorem 1 can be found in Section 5.4. Theorem 1 states that the rank centrality estimator  $\hat{\pi}_*$  achieves an extremal Bayes relative  $\ell^\infty$ -risk of  $\tilde{O}(n^{-1/2})$ , and no other estimator can achieve a risk that decays faster than  $\tilde{\Omega}(n^{-1/2})$ . In the same vein, we show that the *minimax (relative)  $\ell^1$ -risk* (or total variation distance risk) of estimating  $\pi$  based on  $Z$  is also  $\tilde{\Theta}(n^{-1/2})$  (see Table 1).

**THEOREM 2 (Minimax  $\ell^1$ -Risk).** *For sufficiently large constants  $c_{17}, c_{18} > 0$  (which depend on  $\delta$ ,  $p$ , and  $k$ ), and for all sufficiently large  $n \in \mathbb{N}$ :*

$$\frac{c_{17}}{\log(n)\sqrt{n}} \leq \inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}} \mathbb{E}[\|\hat{\pi} - \pi\|_1] \leq \sup_{P_\alpha \in \mathcal{P}} \mathbb{E}[\|\hat{\pi}_* - \pi\|_1] \leq \frac{c_{18}}{\sqrt{n}}.$$

Theorem 2 is established in Section 5.5. The upper bounds in Theorems 1 and 2 follow from (Chen et al. 2019, Theorems 3.1 and 5.2) after some calculations, but the *lower bounds are novel contributions*. We prove them by first lower bounding the minimax risks in terms of Bayes risks in order to circumvent an involved analysis of the infinite-dimensional parameter space  $\mathcal{P}$ . In particular, we set  $P_\alpha \in \mathcal{P}$  to be the uniform PDF over  $[\delta, 1]$ , denoted  $\text{unif}([\delta, 1]) \in \mathcal{P}$ . We then lower bound the Bayes risks using a recent generalization of *Fano's method* (Khas'minskii 1979, Ibragimov and Khas'minskii 1977) (also see (Yu 1997, Tsybakov 2009)), which was specifically developed to produce such lower bounds in the setting where the parameter space is a continuum, e.g.,  $[\delta, 1]$ , instead of a finite set (Zhang 2006, Duchi and Wainwright 2013, Chen et al. 2016, Xu and Raginsky 2017); see Sections 5.2 and 5.3.

The principal analytical difficulty in executing the generalized Fano's method is in deriving a tight upper bound on the *mutual information* between  $\pi$  and  $Z$ , denoted  $I(\pi; Z)$  (see (36) in Section 5.1 for a formal definition), where the probability law of  $\pi$  is defined using  $P_\alpha = \text{unif}([\delta, 1])$ . The ensuing proposition presents our upper bound on  $I(\pi, Z)$ .

**PROPOSITION 2 (Covering Number Bound on Mutual Information).** *For all  $n \geq 2$ , we have:*

$$I(\pi; Z) \leq \frac{1}{2}n \log(n) + \frac{(1-\delta)^2}{8\delta^2} \left(2 + \delta + \frac{1}{\delta}\right) kpn.$$

Proposition 2 is proved in Section 5.1. We note that although standard information inequalities (e.g., Chen et al. 2016, Equation (44)) typically suffice to obtain minimax rates for various estimation problems, they only produce a sub-optimal estimate  $I(\pi; Z) = O(n^2)$  in our problem, as explained at the end of Section 5.1, cf. (50). So, to derive the sharper estimate  $I(\pi; Z) = O(n \log(n))$  in Proposition 2, we execute a careful covering number argument that is inspired by the techniques of Yang and Barron (1999) (also see the distillation in (Wu 2019, Lemma 16.1)).

We make two further remarks. Firstly, it is worth juxtaposing our results with (Chen et al. 2019, Theorem 5.2) and (Negahban et al. 2017, Theorems 2 and 3), which state that the minimax relative  $\ell^2$ -risk of estimating  $\pi$  is  $\Theta(n^{-1/2})$ . This result holds under a worst-case skill parameter value model as opposed to the worst-case prior distribution model of this paper. Secondly, both Theorems 1 and 2 hold verbatim if  $\mathcal{P}$  is replaced by any set of probability measures with support in  $[\delta, 1]$  that contains  $\text{unif}([\delta, 1])$ .

### 3.2. Minimax Bound on Skill PDF Estimation

We now state our main result concerning the estimation error for  $P_\alpha$ . In particular, we argue that the MSE risk of our estimation algorithm (see (9)) scales as  $\tilde{O}(n^{-\eta/(\eta+1)})$  for any  $P_\alpha \in \mathcal{P}$ .

**THEOREM 3 (MSE Upper Bound).** *Fix any sufficiently large constants  $c_2, c_3 > 0$  and suppose that  $p \geq c_2 \log(n)/(\delta^5 n)$ ,  $b \geq c_3 \sqrt{\log(n)/n}$ ,  $\epsilon \geq 5 \log(n)/(bn)$ , and  $\lim_{n \rightarrow \infty} \delta^{-1}(npk)^{-1/2} \log(n)^{1/2} = 0$ . Then, for any  $L_2$ -Lipschitz continuous kernel  $K : [-1, 1] \rightarrow \mathbb{R}$  of order  $\lceil \eta \rceil - 1$ , there exists a sufficiently large constant  $c_{12} > 0$  (that depends on  $\gamma, \eta, B, L_1, L_2$ , and  $K$ ) such that for all sufficiently large  $n \in \mathbb{N}$ :*

$$R_{\text{MSE}}(n) \leq \sup_{P_\alpha \in \mathcal{P}} \mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{\mathcal{P}}^*(x) - P_\alpha(x) \right)^2 dx \right] \leq c_{12} \max \left\{ \left( \frac{1}{\delta^2 pk} \right)^{\frac{\eta}{\eta+1}}, 1 \right\} \left( \frac{\log(n)}{n} \right)^{\frac{\eta}{\eta+1}}.$$

Theorem 3 is established in Section 4.2. We next make several pertinent remarks. Firstly, the condition  $p \geq c_2 \log(n)/(\delta^5 n)$  is precisely the critical scaling that ensures that  $\mathcal{G}(n, p)$  is connected with high probability (cf. Bollobás 2001, Blum et al. 2020, Section 7.1 and Theorem 8.11, respectively). This is essential to estimate  $\alpha_1, \dots, \alpha_n$  in Step 1 of Algorithm 1, since we cannot reasonably compare the skill levels of disconnected players. Secondly, while  $\hat{\mathcal{P}}^*$  can be negative, the non-negative truncated estimator  $\hat{\mathcal{P}}^+(x) = \max\{\hat{\mathcal{P}}^*(x), 0\}$  achieves smaller MSE risk than  $\hat{\mathcal{P}}^*$  (cf. Tsybakov 2009, p.10). So it is easy to construct “good” non-negative estimators. Thirdly, we note that similar analyses to Theorem 3 can be carried out for, e.g., *Nikol’ski* and *Sobolev classes* of PDFs

(cf. [Tsybakov 2009](#), Section 1.2.3). Lastly, it is worth mentioning that BTL models can also be equivalently parametrized using *logit parameters*  $\omega_i = \log(\alpha_i)$ ,  $i \in [n]$ , which are drawn independently from the PDF  $P_\omega(x) = e^x P_\alpha(e^x)$ ,  $x \in \mathbb{R}$ . When  $\delta$  is  $\Theta(1)$ , it can be shown that the MSE of the estimator  $\hat{\mathcal{P}}_{\text{logit}}^*(x) = e^x \hat{\mathcal{P}}^*(e^x)$ ,  $x \in \mathbb{R}$  for  $P_\omega$  is upper bounded by [Theorem 3](#). Therefore, our analysis of [Theorem 3](#) also holds for estimating distributions of logit parameters.

On the other hand, we note that there exists a constant  $c_{13} > 0$  (depending on  $\eta, L_1$ ) such that for all sufficiently large  $n \in \mathbb{N}$ , the following minimax lower bound holds (cf. [Tsybakov 2009](#), [Wasserman 2019](#), Exercise 2.10 and [Theorem 6](#), respectively):

$$R_{\text{MSE}}(n) \geq \inf_{\hat{P}_{\alpha^n}(\cdot)} \sup_{P_\alpha \in \mathcal{P}} \mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{P}_{\alpha^n}(x) - P_\alpha(x) \right)^2 dx \right] \geq c_{13} \left( \frac{1}{n} \right)^{\frac{2\eta}{2\eta+1}} \quad (18)$$

where the infimum is over all estimators  $\hat{P}_{\alpha^n} : \mathbb{R} \rightarrow \mathbb{R}$  of  $P_\alpha$  based on  $\alpha_1, \dots, \alpha_n$ , and the first inequality holds because the infimum in [\(4\)](#) is over a subset of the class of estimators used in the infimum in [\(18\)](#); indeed, given  $\alpha_1, \dots, \alpha_n$ , one can simulate  $Z$  via [\(1\)](#) and estimate  $P_\alpha$  from  $Z$ . Thus, when  $\eta = 1$ , [Theorem 3](#) and [\(18\)](#) show that  $R_{\text{MSE}}(n) = \tilde{O}(n^{-1/2})$  and  $R_{\text{MSE}}(n) = \Omega(n^{-2/3})$ . At the other extreme, if  $P_\alpha$  is smooth, i.e.,  $P_\alpha$  satisfies the boundedness conditions of  $\mathcal{P}$  and is *infinitely differentiable* with all derivatives bounded by  $L_1$ , then it belongs to  $\mathcal{P}(\delta, \epsilon, b, \eta, L_1, B)$  for all  $\eta > 0$ . Hence, [Theorem 3](#) also holds for all  $\eta > 0$  and we may let  $\eta \rightarrow \infty$ . So, for any constant  $\varepsilon > 0$ , letting  $\eta = \varepsilon^{-1} - 1$  yields an  $\tilde{O}(n^{-1+\varepsilon})$  MSE upper bound for such smooth PDFs. Likewise, an  $\Omega(n^{-1})$  minimax lower bound analogous to [\(18\)](#) also holds for smooth PDFs ([Ibragimov and Khas'minskii 1982](#), [Tsybakov 2009](#)). Together, these bounds form the first row of [Table 1](#).

We next emphasize that the key technical step in the proof of [Theorem 3](#) is the ensuing intermediate result.

**PROPOSITION 3 (MSE Decomposition).** *Fix any sufficiently large constants  $c_2, c_3, c_8, c_9 > 0$  and suppose that  $p \geq c_2 \log(n)/(\delta^5 n)$ ,  $b \geq c_3 \sqrt{\log(n)/n}$ ,  $\epsilon \geq 5 \log(n)/(bn)$ , and  $\lim_{n \rightarrow \infty} \delta^{-1} (npk)^{-1/2} \log(n)^{1/2} = 0$ . Then, for any  $P_\alpha \in \mathcal{P}$ , any  $L_2$ -Lipschitz continuous kernel  $K : [-1, 1] \rightarrow \mathbb{R}$ , any bandwidth  $h \in (0, 1]$  satisfying  $h = \Omega(\max\{1/(\delta\sqrt{pk}), 1\} \sqrt{\log(n)/n})$ , and any sufficiently large  $n \in \mathbb{N}$ :*

$$\mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{\mathcal{P}}^*(x) - P_\alpha(x) \right)^2 dx \right] \leq 2 \mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{P}_{\alpha^n}^*(x) - P_\alpha(x) \right)^2 dx \right] + \frac{c_8 B^2 L_2^2}{h^2} \mathbb{E} \left[ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \right] + \frac{c_9 L_2^2}{n^5 h^4}$$

where  $\hat{P}_{\alpha^n}^* : \mathbb{R} \rightarrow \mathbb{R}$  denotes the classical PR kernel density estimator of  $P_\alpha$  ([Rosenblatt 1956](#), [Parzen 1962](#)) based on the true samples  $\alpha_1, \dots, \alpha_n$  (if they were made available by an oracle):

$$\forall x \in \mathbb{R}, \quad \hat{P}_{\alpha^n}^*(x) \triangleq \frac{1}{nh} \sum_{i=1}^n K \left( \frac{\alpha_i - x}{h} \right). \quad (19)$$



The proof of Proposition 3 can be found in Section 4.1. This result decomposes the MSE between  $\widehat{\mathcal{P}}^*$  (with general  $h$ ) and  $P_\alpha$  into two dominant terms: the MSE of estimating  $P_\alpha$  using (19), which can be analyzed using a standard bias-variance tradeoff (see Lemma 2 in Section 4.2 (Tsybakov 2009, Wasserman 2019)), and the squared  $\ell^\infty$ -risk of estimating  $\alpha_1, \dots, \alpha_n$  using (8). To analyze the second term, we use a relative  $\ell^\infty$ -norm bound from (Chen et al. 2019, Theorem 3.1) (see Lemma 1 in Section 4.1); the same bound is also used to obtain the upper bound in Theorem 1.

Finally, as noted at the end of Section 1.1, the ensuing proposition presents an important consequence of Theorem 3.

**PROPOSITION 4 (Estimation of Bounded Statistics).** *Fix any sufficiently large constants  $c_2, c_3 > 0$  and suppose that  $p \geq c_2 \log(n)/(\delta^5 n)$ ,  $b \geq c_3 \sqrt{\log(n)/n}$ ,  $\epsilon \geq 5 \log(n)/(bn)$ , and  $\lim_{n \rightarrow \infty} \delta^{-1}(npk)^{-1/2} \log(n)^{1/2} = 0$ . Then, for any  $L_2$ -Lipschitz continuous kernel  $K : [-1, 1] \rightarrow \mathbb{R}$  of order  $\lceil \eta \rceil - 1$ , there exists a sufficiently large constant  $c_{12} > 0$  (that depends on  $\gamma, \eta, B, L_1, L_2$ , and  $K$ ) such that for all sufficiently large  $n \in \mathbb{N}$ :*

$$\sup_{P_\alpha \in \mathcal{P}} \sup_{f: \mathbb{R} \rightarrow [-1, 1]} \mathbb{E} \left[ \left( \int_{\mathbb{R}} f(x) \widehat{\mathcal{P}}^*(x) dx - \int_{\mathbb{R}} f(x) P_\alpha(x) dx \right)^2 \right] \leq 3 c_{12} \max \left\{ \left( \frac{1}{\delta^2 pk} \right)^{\frac{\eta}{\eta+1}}, 1 \right\} \left( \frac{\log(n)}{n} \right)^{\frac{\eta}{\eta+1}}$$

where the second supremum is over all (Borel measurable) functions  $f : \mathbb{R} \rightarrow [-1, 1]$  that are bounded by 1.

Proposition 4 is derived in Appendix A.2. It conveys that by estimating  $P_\alpha$ , we obtain *uniform guarantees* on estimating any bounded statistic of the form  $\mathbb{E}[f(\alpha)] = \int_{\mathbb{R}} f(x) P_\alpha(x) dx$  (for a bounded function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ) using the estimator  $\int_{\mathbb{R}} f(x) \widehat{\mathcal{P}}^*(x) dx$ . In particular, this implies that we get statistical guarantees on estimating all moments of  $P_\alpha$  since  $P_\alpha$  has support in  $[\delta, 1]$ .

## 4. MSE Upper Bound on Skill Density Estimation

In this section, we will prove Proposition 3 and Theorem 3 in Sections 4.1 and 4.2, respectively.

### 4.1. Proof of Proposition 3

In order to prove Proposition 3, we require the following known result from the literature (Chen et al. 2019), which upper bounds the relative  $\ell^\infty$ -norm loss between  $\widehat{\pi}_*$  and the canonically scaled skill parameters in (15). (As explained earlier in Section 2.3,  $\widehat{\pi}_*$  is intuitively a good estimator of  $\pi$ .)

**LEMMA 1 (Relative  $\ell^\infty$ -Loss Bound (Chen et al. 2019, Theorem 3.1)).** *Suppose that  $p \geq c_2 \log(n)/(\delta^5 n)$  for some sufficiently large constant  $c_2 > 0$ . Then, there exist (universal) constants  $c_4, c_5 > 0$  such that for all sufficiently large  $n \in \mathbb{N}$ , we have:*

$$\mathbb{P} \left( \frac{\|\widehat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \leq \frac{c_4}{\delta} \sqrt{\frac{\log(n)}{npk}} \mid \alpha_1, \dots, \alpha_n \right) \geq 1 - \frac{c_5}{n^5}$$

where the probability is computed with respect to the conditional distribution of the observation matrix  $Z$  and the random graph  $\mathcal{G}(n, p)$  given any realizations of the skill parameters  $\alpha_1, \dots, \alpha_n$ .

We remark that the proof of Lemma 1 in (Chen et al. 2019) crucially uses the assumption that  $\alpha_1, \dots, \alpha_n \in [\delta, 1]$ . We also note that the conditioning on  $\alpha_1, \dots, \alpha_n$  in Lemma 1 reflects the fact that Chen et al. (2019) consider a non-Bayesian scenario where  $\alpha_1, \dots, \alpha_n$  are deterministic (and unknown). In contrast, this work assumes that  $\alpha_1, \dots, \alpha_n$  are drawn i.i.d. from a prior PDF  $P_\alpha$ .

We now proceed to establishing Proposition 3 using Lemma 1.

*Proof of Proposition 3.* We commence this proof with the following bound on the MSE between  $\widehat{\mathcal{P}}^*$  and  $P_\alpha$ :

$$\begin{aligned}
& \mathbb{E} \left[ \int_{\mathbb{R}} \left( \widehat{\mathcal{P}}^*(x) - P_\alpha(x) \right)^2 dx \right] \\
&= \mathbb{E} \left[ \int_{\mathbb{R}} \left( \widehat{\mathcal{P}}^*(x) - \widehat{P}_{\alpha^n}^*(x) + \widehat{P}_{\alpha^n}^*(x) - P_\alpha(x) \right)^2 dx \right] \\
&\leq 2 \mathbb{E} \left[ \int_{\mathbb{R}} \left( \widehat{P}_{\alpha^n}^*(x) - P_\alpha(x) \right)^2 dx \right] + 2 \mathbb{E} \left[ \int_{\mathbb{R}} \left( \widehat{\mathcal{P}}^*(x) - \widehat{P}_{\alpha^n}^*(x) \right)^2 dx \right] \\
&= 2 \mathbb{E} \left[ \int_{\mathbb{R}} \left( \widehat{P}_{\alpha^n}^*(x) - P_\alpha(x) \right)^2 dx \right] + 2 \int_{\mathbb{R}} \mathbb{E} \left[ \left( \widehat{\mathcal{P}}^*(x) - \widehat{P}_{\alpha^n}^*(x) \right)^2 \right] dx \\
&= 2 \mathbb{E} \left[ \int_{\mathbb{R}} \left( \widehat{P}_{\alpha^n}^*(x) - P_\alpha(x) \right)^2 dx \right] + \frac{2}{n^2 h^2} \int_{\mathbb{R}} \mathbb{E} \left[ \left( \sum_{i=1}^n K \left( \frac{\widehat{\alpha}_i - x}{h} \right) - K \left( \frac{\alpha_i - x}{h} \right) \right)^2 \right] dx \\
&= 2 \mathbb{E} \left[ \int_{\mathbb{R}} \left( \widehat{P}_{\alpha^n}^*(x) - P_\alpha(x) \right)^2 dx \right] + \frac{2}{n^2 h^2} \int_{-1}^2 \mathbb{E} \left[ \left( \sum_{i=1}^n K \left( \frac{\widehat{\alpha}_i - x}{h} \right) - K \left( \frac{\alpha_i - x}{h} \right) \right)^2 \right] dx \quad (20)
\end{aligned}$$

where the second inequality follows from the inequality  $(a+b)^2 \leq 2a^2 + 2b^2$  for all  $a, b \in \mathbb{R}$ , the third equality follows from Tonelli's theorem, the fourth equality follows from (9) and (19) (where  $h$  is general, i.e., not necessarily given by (10)), and the fifth equality holds because  $h \leq 1$ ,  $\alpha_i, \widehat{\alpha}_i \in [0, 1]$  for all  $i \in [n]$  (see (8)), and the maps  $\mathbb{R} \ni t \mapsto K((\alpha_i - t)/h)$  and  $\mathbb{R} \ni t \mapsto K((\widehat{\alpha}_i - t)/h)$  have supports contained inside the intervals  $[\alpha_i - h, \alpha_i + h]$  and  $[\widehat{\alpha}_i - h, \widehat{\alpha}_i + h]$ , respectively, for all  $i \in [n]$ . We proceed by bounding the second term in (20).

To this end, we fix any  $x \in \mathbb{R}$ , and define  $S_x \subseteq [n]$  to be the set of players  $i \in [n]$ , whose skill parameters  $\alpha_i$ , or their estimates  $\widehat{\alpha}_i$ , fall into the small (and diminishing) neighborhood  $[x - h, x + h]$ :

$$S_x \triangleq \{i \in [n] : \alpha_i \in [x - h, x + h] \text{ or } \widehat{\alpha}_i \in [x - h, x + h]\}.$$

Then, we may bound the integrand in the second term of (20) as follows:

$$\mathbb{E} \left[ \left( \sum_{i=1}^n K \left( \frac{\widehat{\alpha}_i - x}{h} \right) - K \left( \frac{\alpha_i - x}{h} \right) \right)^2 \right] = \mathbb{E} \left[ \left( \sum_{i \in S_x} K \left( \frac{\widehat{\alpha}_i - x}{h} \right) - K \left( \frac{\alpha_i - x}{h} \right) \right)^2 \right]$$

$$\begin{aligned}
 &\leq \mathbb{E} \left[ \left( \sum_{i \in S_x} \left| K \left( \frac{\hat{\alpha}_i - x}{h} \right) - K \left( \frac{\alpha_i - x}{h} \right) \right| \right)^2 \right] \\
 &\leq \frac{L_2^2}{h^2} \mathbb{E} \left[ \left( \sum_{i \in S_x} |\hat{\alpha}_i - \alpha_i| \right)^2 \right] \\
 &\leq \frac{L_2^2}{h^2} \mathbb{E} \left[ |S_x|^2 \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \right] \tag{21}
 \end{aligned}$$

where the first equality holds because the map  $\mathbb{R} \ni t \mapsto K((t-x)/h)$  has support contained inside the interval  $[x-h, x+h]$ , the second inequality follows from the triangle inequality, and the third inequality follows from the fact that the kernel  $K$  is  $L_2$ -Lipschitz continuous. To further upper bound (21), we now present three claims.

The first claim is a rudimentary auxiliary result that is frequently used in the high-dimensional and non-parametric statistics and theoretical machine learning literature. It says that the intersection of high probability events is itself a high probability event.

**CLAIM 1 (Intersection of High Probability Events).** *Consider any two events  $A_1$  and  $A_2$  with probabilities satisfying  $\mathbb{P}(A_1) \geq 1 - \varepsilon_1$  and  $\mathbb{P}(A_2) \geq 1 - \varepsilon_2$  for any constants  $\varepsilon_1, \varepsilon_2 \in [0, 1]$ . Then, we have:*

$$\mathbb{P}(A_1 \cap A_2) \geq 1 - \varepsilon_1 - \varepsilon_2.$$

*Proof.* This is an immediate corollary of the inclusion-exclusion principle.  $\square$

The second claim utilizes Lemma 1 to show that  $|\hat{\alpha}_i - \alpha_i| = O(\max\{1/(\delta\sqrt{pk}), 1\}\sqrt{\log(n)/n})$  for every  $i \in [n]$  with high probability for all sufficiently large  $n$ .

**CLAIM 2 ( $\ell^\infty$ -Norm Bound on Skill Parameter Estimation).** *There exists a (universal) constant  $c_6 > 0$  such that for all sufficiently large  $n \in \mathbb{N}$ :*

$$\mathbb{P} \left( \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i| \leq c_6 \max \left\{ \frac{1}{\delta\sqrt{pk}}, 1 \right\} \sqrt{\frac{\log(n)}{n}} \right) \geq 1 - \frac{c_5 + 1}{n^5}$$

where  $c_5 > 0$  is the fixed constant from Lemma 1.

*Proof.* We prove this claim in four steps. Firstly, we establish that for all sufficiently large  $n \in \mathbb{N}$ :

$$\mathbb{P} \left( \max_{i \in [n]} \alpha_i \geq 1 - \frac{5 \log(n)}{bn} \right) \geq 1 - \frac{1}{n^5}. \tag{22}$$

To establish (22), note that:

$$\begin{aligned}
 \mathbb{P} \left( \forall i \in [n], \alpha_i < 1 - \frac{5 \log(n)}{bn} \right) &= \mathbb{P} \left( \alpha_1 < 1 - \frac{5 \log(n)}{bn} \right)^n \\
 &= \left( 1 - \int_{1-5 \log(n)/(bn)}^1 P_\alpha(t) dt \right)^n
 \end{aligned}$$

$$\begin{aligned}
&\leq \left(1 - \frac{5 \log(n)}{n}\right)^n \\
&= \exp\left(n \log\left(1 - \frac{5 \log(n)}{n}\right)\right) \\
&\leq \frac{1}{n^5}
\end{aligned}$$

where the first equality holds because  $\alpha_1, \dots, \alpha_n$  are i.i.d., the third inequality holds because we have assumed that  $\epsilon \geq 5 \log(n)/(bn)$  and  $P_\alpha$  satisfies the lower bound  $P_\alpha(t) \geq b$  for all  $t \in [1 - \epsilon, 1]$ , and for every large enough  $n$ , the fifth inequality follows from the well-known bound  $\log(1 + x) \leq x$  for all  $x > -1$ . This produces the desired bound (22).

Secondly, we define the normalized skill parameter random variables:

$$\forall i \in [n], \tilde{\alpha}_i \triangleq \frac{\alpha_i}{\max_{j \in [n]} \alpha_j} = \frac{\pi(i)}{\|\pi\|_\infty}$$

where  $\pi$  denotes the probability vector of canonically scaled skill parameters in (15). It turns out that for all sufficiently large  $n \in \mathbb{N}$ :

$$\mathbb{P}\left(\forall i \in [n], 0 \leq \tilde{\alpha}_i - \alpha_i \leq \frac{10}{c_3} \sqrt{\frac{\log(n)}{n}}\right) \geq 1 - \frac{1}{n^5} \quad (23)$$

i.e., the normalized skill parameters are close to the true skill parameters with high probability. To derive (23), suppose that the event in (22) occurs. Then, we have:

$$1 - \frac{5 \log(n)}{bn} \leq \max_{i \in [n]} \alpha_i \leq 1$$

which implies that for every  $i \in [n]$ :

$$\alpha_i \leq \tilde{\alpha}_i \leq \alpha_i \left(1 - \frac{5 \log(n)}{bn}\right)^{-1} \leq \alpha_i \left(1 - \frac{5}{c_3} \sqrt{\frac{\log(n)}{n}}\right)^{-1} \leq \alpha_i \left(1 + \frac{10}{c_3} \sqrt{\frac{\log(n)}{n}}\right)$$

where the third inequality follows from the lower bound we have assumed on  $b$  in the proposition statement, and the fourth inequality holds for all sufficiently large  $n \in \mathbb{N}$ , because the bound  $(1 - x)^{-1} = 1 + x + x^2(1 - x)^{-1} \leq 1 + x + 2x^2 \leq 1 + 2x$  holds for any  $x \in (0, \frac{1}{2}]$ . Thus, since  $\alpha_i \leq 1$  for all  $i \in [n]$ , we get (23).

Thirdly, we approximate the normalized skill parameters  $\tilde{\alpha}_i$  using the estimates  $\hat{\alpha}_i$ . Recall from Lemma 1 that for all sufficiently large  $n \in \mathbb{N}$ :

$$\mathbb{P}\left(\frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \leq \frac{c_4}{\delta} \sqrt{\frac{\log(n)}{npk}}\right) \geq 1 - \frac{c_5}{n^5} \quad (24)$$

where we take expectations with respect to the law of  $\alpha_1, \dots, \alpha_n$ . Suppose that the event in (24) happens. Then, for any  $i \in [n]$ , notice that for all sufficiently large  $n \in \mathbb{N}$ :

$$\hat{\alpha}_i = \frac{\hat{\pi}_*(i)}{\|\hat{\pi}_*\|_\infty}$$

$$\begin{aligned}
&= \frac{\pi(i) + (\hat{\pi}_*(i) - \pi(i))}{\|\pi\|_\infty + (\|\hat{\pi}_*\|_\infty - \|\pi\|_\infty)} \\
&\leq \frac{\pi(i) + \|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty - \|\hat{\pi}_*\|_\infty - \|\pi\|_\infty} \\
&\leq \frac{\pi(i) + \|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty - \|\hat{\pi}_* - \pi\|_\infty} \\
&= \left( \tilde{\alpha}_i + \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \right) \left( 1 - \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \right)^{-1} \\
&\leq \left( \tilde{\alpha}_i + \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \right) \left( 1 + 2 \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \right) \\
&\leq \tilde{\alpha}_i + 4 \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty}
\end{aligned}$$

where the first equality follows from (8), the fourth inequality follows from the (reverse) Minkowski inequality, the sixth inequality holds for all sufficiently large  $n \in \mathbb{N}$  because: 1) the event in (24) occurs, 2) we have assumed that  $\lim_{n \rightarrow \infty} \delta^{-1}(npk)^{-1/2} \log(n)^{1/2} = 0$  in the proposition statement, and 3) we use the bound  $(1-x)^{-1} \leq 1+2x$  for  $x \in (0, \frac{1}{2}]$ , and the seventh inequality holds because  $\tilde{\alpha}_i \leq 1$  for all  $i \in [n]$ . Likewise, using analogous reasoning, observe that for all sufficiently large  $n \in \mathbb{N}$ :

$$\begin{aligned}
\hat{\alpha}_i &\geq \frac{\pi(i) - \|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty + \|\hat{\pi}_* - \pi\|_\infty} \\
&= \left( \tilde{\alpha}_i - \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \right) \left( 1 + \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \right)^{-1} \\
&\geq \left( \tilde{\alpha}_i - \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \right) \left( 1 - \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \right) \\
&\geq \tilde{\alpha}_i - 2 \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty}.
\end{aligned}$$

Therefore, using Lemma 1, we obtain that for all sufficiently large  $n \in \mathbb{N}$ :

$$\mathbb{P} \left( \max_{i \in [n]} |\hat{\alpha}_i - \tilde{\alpha}_i| \leq 4 \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \right) \geq 1 - \frac{c_5}{n^5}. \quad (25)$$

Finally, we combine (23) and (25) together. Suppose that the events in both (23) and (24) occur. Then, we get:

$$\begin{aligned}
\max_{i \in [n]} |\hat{\alpha}_i - \alpha_i| &\leq \max_{i \in [n]} |\hat{\alpha}_i - \tilde{\alpha}_i| + \max_{i \in [n]} |\tilde{\alpha}_i - \alpha_i| \\
&\leq 4 \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} + \frac{10}{c_3} \sqrt{\frac{\log(n)}{n}} \\
&\leq \frac{4c_4}{\delta} \sqrt{\frac{\log(n)}{npk}} + \frac{10}{c_3} \sqrt{\frac{\log(n)}{n}} \\
&= O \left( \max \left\{ \frac{1}{\delta \sqrt{pk}}, 1 \right\} \sqrt{\frac{\log(n)}{n}} \right)
\end{aligned}$$

where the first inequality follows from the triangle inequality, the second inequality follows from (23) and (25), and the third inequality follows from (24) (or Lemma 1). Using Claim 1, this produces the desired bound:

$$\mathbb{P}\left(\max_{i \in [n]} |\hat{\alpha}_i - \alpha_i| = O\left(\max\left\{\frac{1}{\delta\sqrt{pk}}, 1\right\}\sqrt{\frac{\log(n)}{n}}\right)\right) \geq 1 - \frac{c_5 + 1}{n^5}$$

for all sufficiently large  $n$ .  $\square$

Finally, our third claim uses Claim 2 to argue that with high probability, the cardinality of  $S_x$  is not too large.

**CLAIM 3 (Cardinality Bound for  $S_x$ ).** *There exists a sufficiently large (universal) constant  $c_7 > 0$  such that for every sufficiently large  $n \in \mathbb{N}$ , we have:*

$$\mathbb{P}(|S_x| \leq c_7 B h n) \geq 1 - \frac{c_5 + 2}{n^5}.$$

*Proof.* First, for any constant  $\tau > 1$  (to be chosen later), we define  $N_h \in [n] \cup \{0\}$  to be the discrete random variable representing the number of players  $i \in [n]$  for whom  $\alpha_i$  belongs to the interval  $[x - \tau h, x + \tau h]$ :

$$N_h \triangleq \sum_{i=1}^n \mathbb{1}\{\alpha_i \in [x - \tau h, x + \tau h]\}.$$

Then, fix any  $\varepsilon \geq 0$ , and observe using Lemma 10 in Appendix B that:

$$\mathbb{P}\left(\frac{1}{n}N_h - \mathbb{P}(\alpha_1 \in [x - \tau h, x + \tau h]) > \varepsilon\right) \leq \exp(-2n\varepsilon^2)$$

which uses the fact that  $\{\mathbb{1}\{\alpha_i \in [x - \tau h, x + \tau h]\} : i \in [n]\}$  are i.i.d. Bernoulli random variables with mean  $\mathbb{P}(\alpha_1 \in [x - \tau h, x + \tau h])$ , since  $\alpha_1, \dots, \alpha_n$  are drawn i.i.d. from  $P_\alpha$ . At this point, letting  $\varepsilon = \sqrt{5 \log(n)/(2n)}$  yields:

$$\mathbb{P}\left(\frac{1}{n}N_h - \mathbb{P}(\alpha_1 \in [x - \tau h, x + \tau h]) > \sqrt{\frac{5 \log(n)}{2n}}\right) \leq \exp\left(-\frac{5n \log(n)}{n}\right) = \frac{1}{n^5}. \quad (26)$$

Next, recall that  $P_\alpha \in \mathcal{P}$  is uniformly bounded (almost everywhere) by  $B > 0$ , i.e.,  $P_\alpha(t) \leq B$  for all  $t \in \mathbb{R}$ . Using this bound, we obtain:

$$\mathbb{P}(\alpha_1 \in [x - \tau h, x + \tau h]) = \int_{x - \tau h}^{x + \tau h} P_\alpha(t) dt \leq 2B\tau h. \quad (27)$$

Hence, combining (26) and (27), we have that with probability at least  $1 - n^{-5}$ :

$$\begin{aligned} \frac{1}{n}N_h &\leq \mathbb{P}(\alpha_1 \in [x - \tau h, x + \tau h]) + \sqrt{\frac{5 \log(n)}{2n}} \\ &\leq 2B\tau h + \sqrt{\frac{5 \log(n)}{2n}}. \end{aligned}$$

Equivalently, we have derived the following bound:

$$\mathbb{P}\left(N_h \leq 2B\tau hn + \sqrt{\frac{5}{2}n \log(n)}\right) \geq 1 - \frac{1}{n^5}. \quad (28)$$

Now, recall from the proposition statement that  $h = \Omega(\max\{1/(\delta\sqrt{pk}), 1\}\sqrt{\log(n)/n})$ . Hence, we may choose  $\tau > 1$  large enough so that for all sufficiently large  $n$ , we have:

$$(\tau - 1)h \geq c_6 \max\left\{\frac{1}{\delta\sqrt{pk}}, 1\right\} \sqrt{\frac{\log(n)}{n}} \quad (29)$$

where  $c_6 > 0$  is the constant from Claim 2. Assume that both the events in (28) and Claim 2 occur. Then, for any  $i \in [n]$ , if  $\alpha_i \in [x - h, x + h]$ , then we trivially have  $\alpha_i \in [x - \tau h, x + \tau h]$  since  $\tau > 1$ . On the other hand, if  $\hat{\alpha}_i \in [x - h, x + h]$ , then we have  $\alpha_i \in [x - \tau h, x + \tau h]$ , because  $|\hat{\alpha}_i - \alpha_i| \leq c_6 \max\{1/(\delta\sqrt{pk}), 1\} \sqrt{\log(n)/n} \leq (\tau - 1)h$  by Claim 2 and (29). Thus, we get:

$$|S_x| \leq N_h$$

with  $\tau$  chosen according to (29). Applying Claim 1, Claim 2, and (28) together yields:

$$\mathbb{P}\left(|S_x| \leq 2B\tau hn + \sqrt{\frac{5}{2}n \log(n)}\right) \geq 1 - \frac{c_5 + 2}{n^5}$$

for every sufficiently large  $n$ . Lastly, let  $c_7 > 2\tau$  be a sufficiently large constant so that for all sufficiently large  $n$ :

$$(c_7 - 2\tau)Bh \geq \sqrt{\frac{5 \log(n)}{2n}}$$

which is consistent with our assumption that  $h = \Omega(\max\{1/(\delta\sqrt{pk}), 1\}\sqrt{\log(n)/n})$ . Therefore, we may write:

$$\mathbb{P}(|S_x| \leq c_7 Bhn) \geq 1 - \frac{c_5 + 2}{n^5}$$

for every sufficiently large  $n$ , which completes the proof.  $\square$

Having developed Claim 3, we are now in a position to upper bound (21). For any sufficiently large  $n$ , define the event in Claim 3 as:

$$A_n \triangleq \{|S_x| \leq c_7 Bhn\}.$$

Then, observe that:

$$\begin{aligned} & \mathbb{E}\left[|S_x|^2 \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2\right] \\ &= \mathbb{E}\left[|S_x|^2 \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \middle| A_n\right] \mathbb{P}(A_n) + \mathbb{E}\left[|S_x|^2 \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \middle| A_n^c\right] \mathbb{P}(A_n^c) \\ &\leq c_7^2 B^2 h^2 n^2 \mathbb{E}\left[\max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \middle| A_n\right] \mathbb{P}(A_n) + \frac{c_5 + 2}{n^3} \end{aligned}$$

$$\begin{aligned}
&\leq c_7^2 B^2 h^2 n^2 \left( \mathbb{E} \left[ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \middle| A_n \right] \mathbb{P}(A_n) + \underbrace{\mathbb{E} \left[ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \middle| A_n^c \right] \mathbb{P}(A_n^c)}_{\geq 0} \right) + \frac{c_5 + 2}{n^3} \\
&= c_7^2 B^2 h^2 n^2 \mathbb{E} \left[ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \right] + \frac{c_5 + 2}{n^3} \tag{30}
\end{aligned}$$

where the first and fourth equalities follow from the tower property, and the second inequality uses Claim 3 and the facts that  $|S_x| \leq n$  and  $|\hat{\alpha}_i - \alpha_i| \leq 1$  for all  $i \in [n]$  (see (8)). Plugging (30) into (21) produces:

$$\mathbb{E} \left[ \left( \sum_{i=1}^n K \left( \frac{\hat{\alpha}_i - x}{h} \right) - K \left( \frac{\alpha_i - x}{h} \right) \right)^2 \right] \leq \frac{L_2^2}{h^2} \left( c_7^2 B^2 h^2 n^2 \mathbb{E} \left[ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \right] + \frac{c_5 + 2}{n^3} \right).$$

We can then substitute this bound into (20) to obtain:

$$\begin{aligned}
&\mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{\mathcal{P}}^*(x) - P_\alpha(x) \right)^2 dx \right] \\
&\leq 2 \mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{P}_{\alpha_n}^*(x) - P_\alpha(x) \right)^2 dx \right] + \frac{2L_2^2}{n^2 h^4} \int_{-1}^2 c_7^2 B^2 h^2 n^2 \mathbb{E} \left[ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \right] + \frac{c_5 + 2}{n^3} dx \\
&= 2 \mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{P}_{\alpha_n}^*(x) - P_\alpha(x) \right)^2 dx \right] + \frac{6L_2^2}{n^2 h^4} \left( c_7^2 B^2 h^2 n^2 \mathbb{E} \left[ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \right] + \frac{c_5 + 2}{n^3} \right) \\
&= 2 \mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{P}_{\alpha_n}^*(x) - P_\alpha(x) \right)^2 dx \right] + \frac{6c_7^2 B^2 L_2^2}{h^2} \mathbb{E} \left[ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \right] + \frac{6(c_5 + 2)L_2^2}{n^5 h^4}
\end{aligned}$$

for all sufficiently large  $n$ . This completes the proof after letting  $c_8 = 6c_7^2$  and  $c_9 = 6(c_5 + 2)$ .  $\square$

## 4.2. Proof of Theorem 3

To establish Theorem 3, we first present a well-known lemma which conveys the tradeoff between the bias and variance of the classical PR kernel density estimator  $\hat{P}_{\alpha_n}^*$  defined in (19) (cf. Tsybakov 2009, Wasserman 2019, Propositions 1.2 & 1.4 and Lemmata 3 & 4, respectively).

**LEMMA 2 (Bias-Variance Tradeoff (Tsybakov 2009, Wasserman 2019)).** *For any  $P_\alpha \in \mathcal{P}$ , any kernel  $K : [-1, 1] \rightarrow \mathbb{R}$  of order  $s = \lceil \eta \rceil - 1$ , any  $n \in \mathbb{N}$ , and any bandwidth  $h \in (0, 1]$ , we have:*

$$\mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{P}_{\alpha_n}^*(x) - P_\alpha(x) \right)^2 dx \right] \leq \frac{1}{nh} \left( \int_{-1}^1 K(x)^2 dx \right) + 3h^{2\eta} \left( \frac{L_1}{s!} \int_{-1}^1 |x|^\eta |K(x)| dx \right)^2$$

where  $\alpha_1, \dots, \alpha_n$  are i.i.d. with distribution  $P_\alpha$ .

In Lemma 2, it is well-known that the first term captures the variance of  $\hat{P}_{\alpha_n}^*$ , and the second term bounds the squared bias of  $\hat{P}_{\alpha_n}^*$ . Specifically, as shown in (Tsybakov 2009, Wasserman 2019), the bound on the variance term uses the property that the kernel is square-integrable, and the bound on the bias term uses the other properties in the definition of a kernel as well as the Hölder class assumption on  $P_\alpha$  (outlined earlier). Furthermore, we remark that the bound on the bias term



in Lemma 2 follows from (Tsybakov 2009, Proposition 1.2) by noting that  $P_\alpha$  has its support in the interval  $[0, 1]$  and  $\hat{P}_{\alpha^n}^*$  has its support in the interval  $[-1, 2]$  (because the kernel  $K$  has its support in  $[-1, 1]$  and the bandwidth  $h \leq 1$ ). In fact, the length of the interval  $[0, 1] \cup [-1, 2] = [-1, 2]$  is what gives rise to the constant 3 in Lemma 2.

We next prove Theorem 3 using Lemmata 1 and 2 and Proposition 3.

*Proof of Theorem 3.* We begin by recalling the result of Proposition 3. There exist sufficiently large constants  $c_8, c_9 > 0$  such that for all sufficiently large  $n \in \mathbb{N}$ :

$$\mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{\mathcal{P}}^*(x) - P_\alpha(x) \right)^2 dx \right] \leq 2 \mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{P}_{\alpha^n}^*(x) - P_\alpha(x) \right)^2 dx \right] + \frac{c_8 B^2 L_2^2}{h^2} \mathbb{E} \left[ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \right] + \frac{c_9 L_2^2}{n^5 h^4}$$

where we assume that the bandwidth  $h \in (0, 1]$  satisfies  $h = \Omega(\max\{1/(\delta\sqrt{pk}), 1\}\sqrt{\log(n)/n})$  and that  $\lim_{n \rightarrow \infty} \delta^{-1}(npk)^{-1/2} \log(n)^{1/2} = 0$ . Now define the constants:

$$c_{10} = 2 \int_{-1}^1 K(x)^2 dx,$$

$$c_{11} = 6 \left( \frac{L_1}{s!} \int_{-1}^1 |x|^\eta |K(x)| dx \right)^2,$$

which depend on the parameters  $\eta$  and  $L_1$  (that define the non-parametric class of PDFs  $\mathcal{P}$ ) and on the kernel  $K : [-1, 1] \rightarrow \mathbb{R}$ . Then, applying Lemma 2 to the first term of the inequality in Proposition 3, we obtain:

$$\mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{\mathcal{P}}^*(x) - P_\alpha(x) \right)^2 dx \right] \leq \frac{c_{10}}{nh} + c_{11} h^{2\eta} + \frac{c_8 B^2 L_2^2}{h^2} \mathbb{E} \left[ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \right] + \frac{c_9 L_2^2}{n^5 h^4} \quad (31)$$

for all sufficiently large  $n$ . We next upper bound the  $\mathbb{E}[\max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2]$  term in (31). To this end, define the event:

$$A \triangleq \left\{ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i| \leq c_6 \max \left\{ \frac{1}{\delta\sqrt{pk}}, 1 \right\} \sqrt{\frac{\log(n)}{n}} \right\}$$

using the constant  $c_6 > 0$  from Claim 2 in the proof of Proposition 3 in Section 4.1, and recall from Claim 2 that there exist constants  $c_5, c_6 > 0$  such that for all sufficiently large  $n \in \mathbb{N}$ :

$$\mathbb{P}(A) \geq 1 - \frac{c_5 + 1}{n^5}. \quad (32)$$

Now observe that for all sufficiently large  $n$ :

$$\begin{aligned} \mathbb{E} \left[ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \right] &= \mathbb{E} \left[ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \middle| A \right] \mathbb{P}(A) + \mathbb{E} \left[ \max_{i \in [n]} |\hat{\alpha}_i - \alpha_i|^2 \middle| A^c \right] \mathbb{P}(A^c) \\ &\leq c_6^2 \max \left\{ \frac{1}{\delta^2 pk}, 1 \right\} \frac{\log(n)}{n} + \frac{c_5 + 1}{n^5} \\ &\leq 2c_6^2 \max \left\{ \frac{1}{\delta^2 pk}, 1 \right\} \frac{\log(n)}{n} \end{aligned} \quad (33)$$

where the first equality uses the law of total expectation, the second inequality follows from (32) and the fact that  $|\hat{\alpha}_i - \alpha_i| \leq 1$  for all  $i \in [n]$  (see (8)), and the final inequality holds for all sufficiently large  $n$ .

Substituting (33) into (31) produces:

$$\mathbb{E} \left[ \int_{\mathbb{R}} \left( \hat{\mathcal{P}}^*(x) - P_\alpha(x) \right)^2 dx \right] \leq \frac{c_{10}}{nh} + c_{11}h^{2\eta} + \frac{2c_8c_6^2B^2L_2^2}{h^2} \max \left\{ \frac{1}{\delta^2pk}, 1 \right\} \frac{\log(n)}{n} + \frac{c_9L_2^2}{n^5h^4} \quad (34)$$

for all sufficiently large  $n$ . All that remains is to minimize this bound over the choice of  $h$  and show that (10) provides the optimal bound. Since the first three terms on the right hand side of (34) will dominate the fourth term, we focus on optimizing these three terms. Notice that the second term is monotone increasing in  $h$ , while the first and third terms are monotone decreasing in  $h$ . So, the optimal scaling of  $h$  with  $n$  can be obtained by either balancing the first and second terms, or balancing the second and third terms. To decide which pair of terms to balance, let us temporarily neglect  $\delta$ ,  $p$ , and  $k$ . Then, if we balance the first and second terms, we get (see, e.g., [Tsybakov 2009](#), Section 1.2.3):

$$\frac{c_{10}}{nh} = c_{11}h^{2\eta} \quad \Leftrightarrow \quad h = \Theta \left( n^{-\frac{1}{2\eta+1}} \right)$$

which implies that the right hand side of (34) is  $\tilde{\Theta} \left( n^{-(2\eta-1)/(2\eta+1)} \right)$ . On the other hand, if we balance the second and third terms, we get:

$$c_{11}h^{2\eta} = \frac{2c_8c_6^2B^2L_2^2}{h^2} \max \left\{ \frac{1}{\delta^2pk}, 1 \right\} \frac{\log(n)}{n} \quad \Leftrightarrow \quad h = \Theta \left( \max \left\{ \frac{1}{\delta^{\frac{1}{\eta+1}}(pk)^{\frac{1}{2\eta+2}}}, 1 \right\} \left( \frac{\log(n)}{n} \right)^{\frac{1}{2\eta+2}} \right)$$

which implies that the right hand side of (34) is  $\tilde{\Theta} \left( n^{-\eta/(\eta+1)} \right)$ . Since  $\frac{\eta}{\eta+1} > \frac{2\eta-1}{2\eta+1}$  for all  $\eta > 0$ , balancing the second and third terms yields the tighter bound on the right hand side of (34).

We conclude this proof by explicitly balancing the second and third terms, and computing the precise resulting expression on the right hand side of (34). For any constant  $\gamma > 0$ , let the bandwidth be as defined in (10):

$$h = \gamma \max \left\{ \frac{1}{\delta^{\frac{1}{\eta+1}}(pk)^{\frac{1}{2\eta+2}}}, 1 \right\} \left( \frac{\log(n)}{n} \right)^{\frac{1}{2\eta+2}}.$$

(Note that it is straightforward to verify that the condition  $h = \Omega(\max\{1/(\delta\sqrt{pk}), 1\}\sqrt{\log(n)/n})$  is satisfied by (10). Indeed, since we know that  $\lim_{n \rightarrow \infty} \delta^{-1}(npk)^{-1/2} \log(n)^{1/2} = 0$ , we also have  $\lim_{n \rightarrow \infty} \max\{1/(\delta\sqrt{pk}), 1\}\sqrt{\log(n)/n} = 0$ . So, we must have  $\max\{1/(\delta\sqrt{pk}), 1\}\sqrt{\log(n)/n} = O((\max\{1/(\delta\sqrt{pk}), 1\}\sqrt{\log(n)/n})^{1/(\eta+1)})$ .) Then, the terms on the right hand side of (34) can be written as:

$$\begin{aligned} \frac{c_{10}}{nh} &= \frac{c_{10}}{\gamma} \min \left\{ \delta^{\frac{1}{\eta+1}}(pk)^{\frac{1}{2\eta+2}}, 1 \right\} \log(n)^{-\frac{1}{2\eta+2}} n^{-\frac{2\eta+1}{2\eta+2}}, \\ c_{11}h^{2\eta} &= c_{11}\gamma^{2\eta} \max \left\{ \delta^{-\frac{2\eta}{\eta+1}}(pk)^{-\frac{\eta}{\eta+1}}, 1 \right\} \log(n)^{\frac{\eta}{\eta+1}} n^{-\frac{\eta}{\eta+1}}, \end{aligned}$$

$$\begin{aligned} \frac{2c_8c_6^2B^2L_2^2}{h^2} \max\left\{\frac{1}{\delta^2pk}, 1\right\} \frac{\log(n)}{n} &= \frac{2c_8c_6^2B^2L_2^2}{\gamma^2} \max\left\{\delta^{-\frac{2\eta}{\eta+1}}(pk)^{-\frac{\eta}{\eta+1}}, 1\right\} \log(n)^{\frac{\eta}{\eta+1}} n^{-\frac{\eta}{\eta+1}}, \\ \frac{c_9L_2^2}{n^5h^4} &= \frac{c_9L_2^2}{\gamma^4} \min\left\{\delta^{\frac{4}{\eta+1}}(pk)^{\frac{2}{\eta+1}}, 1\right\} \log(n)^{-\frac{2}{\eta+1}} n^{-\frac{5\eta+3}{\eta+1}}. \end{aligned}$$

Clearly, the second and third terms are balanced, and dominate the first and fourth terms on the right hand side of (34) as  $n$  grows. Since  $\gamma$ ,  $\eta$ ,  $B$ , and  $L_2$  are constant parameters that do not depend on  $n$ , we have that for all sufficiently large  $n \in \mathbb{N}$ :

$$\mathbb{E}\left[\int_{\mathbb{R}} \left(\widehat{\mathcal{P}}^*(x) - P_\alpha(x)\right)^2 dx\right] \leq c_{12} \max\left\{\left(\frac{1}{\delta^2pk}\right)^{\frac{\eta}{\eta+1}}, 1\right\} \left(\frac{\log(n)}{n}\right)^{\frac{\eta}{\eta+1}}$$

where  $c_{12} > 0$  is a sufficiently large constant that depends on  $\gamma$ ,  $\eta$ ,  $B$ ,  $L_1$ ,  $L_2$ , and the kernel  $K$ . This completes the proof.  $\square$

## 5. Minimax Lower Bounds via Generalized Fano's Method

In this section, we establish the minimax bounds in Theorems 1 and 2. In order to simplify the exposition, we first establish the upper bound on  $I(\pi; Z)$  in Proposition 2 in Section 5.1, present the generalized Fano's method in Section 5.2, derive some useful auxiliary lemmata in Section 5.3, and then present the proofs of Theorems 1 and 2 in Sections 5.4 and 5.5, respectively.

### 5.1. Mutual Information, Covering Numbers, and the Proof of Proposition 2

In order to prove Proposition 2, we commence by presenting some basic definitions and properties of mutual information from information theory. Recall that for any two probability measures  $\mu$  and  $\nu$  over the same measurable space  $(\Omega, \mathcal{F})$ , the *Kullback-Leibler (KL) divergence* (or relative entropy) of  $\nu$  from  $\mu$  is defined as (cf. Polyanskiy and Wu 2017, Definition 1.4):

$$D(\mu||\nu) \triangleq \begin{cases} \int_{\Omega} \log\left(\frac{d\mu}{d\nu}\right) d\mu, & \mu \text{ is absolutely continuous with respect to } \nu \\ +\infty, & \text{otherwise} \end{cases} \quad (35)$$

where  $\frac{d\mu}{d\nu}$  is the Radon-Nikodym derivative (or density) of  $\mu$  with respect to  $\nu$ . Using (35), for any pair of jointly distributed random variables  $X$  and  $Y$ , we define the *mutual information* between  $X$  and  $Y$  as (cf. Polyanskiy and Wu 2017, Definition 2.3):

$$I(X; Y) \triangleq D(P_{X,Y}||P_X \otimes P_Y) \quad (36)$$

where  $P_{X,Y}$  denotes the joint probability law of  $X$  and  $Y$ , and  $P_X \otimes P_Y$  denotes the product measure of the corresponding marginal probability laws of  $X$  and  $Y$ , respectively. Note that in (36), the random variables  $X$  or  $Y$  can be compound variables. So, for example, if  $Y = (Y_1, Y_2)$ , i.e.,  $Y$  is actually a pair of random variables  $Y_1, Y_2$  (where  $X, Y_1, Y_2$  are jointly distributed), then we can

write  $I(X; Y) = I(X; Y_1, Y_2)$ . Furthermore, given three jointly distributed random variables  $X, Y, W$ , where  $W \in \mathcal{W}$  is a discrete random variable whose probability distribution  $P_W$  has support  $\mathcal{W}$ , we define the mutual information between  $X$  and  $Y$  given  $W = w$  as:

$$\forall w \in \mathcal{W}, \quad I(X; Y|W = w) \triangleq D(P_{X,Y|W=w} \| P_{X|W=w} \otimes P_{Y|W=w}) \quad (37)$$

where  $P_{X,Y|W=w}$  denotes the conditional (joint) probability law of  $X$  and  $Y$  given  $W = w$ , and  $P_{X|W=w} \otimes P_{Y|W=w}$  denotes the product measure of the conditional (marginal) probability laws of  $X$  and  $Y$  given  $W = w$ , respectively. Then, using (37), the *conditional mutual information* between  $X$  and  $Y$  given  $W$  is defined as (cf. Polyanskiy and Wu 2017, Definition 2.4):

$$I(X; Y|W) \triangleq \sum_{w \in \mathcal{W}} P_W(w) I(X; Y|W = w) \quad (38)$$

which the expected value of  $I(X; Y|W = w)$  with respect to  $P_W$ . We will utilize the following well-known properties of mutual information in the sequel.

**LEMMA 3 (Properties of Mutual Information).** *For any three jointly distributed random variables  $X, Y, W$ , the following results hold:*

1. (Chain rule (Polyanskiy and Wu 2017, Theorem 2.5), (Cover and Thomas 2006, Theorem 2.5.2)) *If  $W \in \mathcal{W}$  is a discrete random variable whose probability distribution  $P_W$  has support  $\mathcal{W}$ , then:*

$$I(X; Y, W) = I(X; W) + I(X; Y|W).$$

2. (Data processing inequality (Polyanskiy and Wu 2017, Theorem 2.5), (Cover and Thomas 2006, Theorem 2.8.1)) *If  $X \rightarrow Y \rightarrow W$  forms a Markov chain, i.e.,  $X$  and  $W$  are conditionally independent given  $Y$ , then:*

$$I(X; W) \leq I(Y; W).$$

Another set of ideas we will exploit to prove Proposition 2 concerns a powerful and general approach to upper bound mutual information (or more generally, Shannon capacity (cf. Polyanskiy and Wu 2017, Cover and Thomas 2006, Sections 4.4–4.5 and Chapter 7, respectively)) via covering arguments. While there are several variants of such arguments in the literature, in this paper, we will resort to the classical covering argument of Yang and Barron (1999). (We refer readers to (Chen et al. 2016, Section 5) for generalizations of such covering arguments for a class of  $f$ -informativities (Csiszár 1972).)

Recall the formal setup of our problem in Sections 1 and 2. To present the technique in (Yang and Barron 1999), let us condition on any fixed realization of the underlying Erdős-Rényi random

graph  $\mathcal{G}(n, p) = G$ . Then, the partial observations  $Z$ , defined in (3), can be equivalently represented using the (compound) random variable:

$$Z_G \triangleq \{Z(i, j) : \{i, j\} \in G \text{ for } i, j \in [n] \text{ with } i < j\} \quad (39)$$

where the notation  $\{i, j\} \in G$  shows that the undirected edge  $\{i, j\}$  exists in the graph  $G$ , and each  $Z(i, j) = \frac{1}{k} \sum_{m=1}^k Z_m(i, j)$  given  $\{i, j\} \in G$  (where the likelihoods of the  $Z_m(i, j)$ 's are defined via (1)). Moreover, using (39), we have the relation:

$$I(\alpha_1, \dots, \alpha_n; Z | \mathcal{G}(n, p) = G) = I(\alpha_1, \dots, \alpha_n; Z_G) \quad (40)$$

where we use (37), and the fact that the random variables  $\alpha_1, \dots, \alpha_n, Z_G$  are independent of  $\mathcal{G}(n, p)$ . (Note that if  $G$  contains no edges, then  $Z_G$  is a deterministic quantity and  $I(\alpha_1, \dots, \alpha_n; Z_G) = 0$ .) Now consider the  $n$ -dimensional hypercube  $[\delta, 1]^n$  in which the skill parameter random variables  $(\alpha_1, \dots, \alpha_n)$  take values. For any parameter vector (realization)  $\beta = (\beta_1, \dots, \beta_n) \in [\delta, 1]^n$ , let  $P_{Z_G | \beta}$  denote the conditional probability distribution of  $Z_G$  given  $\alpha_i = \beta_i$  for all  $i \in [n]$  (with abuse of notation); see (1) and (3). Then, for any  $\varepsilon > 0$ , we define an  $\varepsilon$ -covering of  $[\delta, 1]^n$  with finite cardinality  $M \in \mathbb{N}$  to be a subset of parameter vectors  $\{\beta^{(1)}, \dots, \beta^{(M)}\} \subset [\delta, 1]^n$  that satisfies:

$$\forall \beta \in [\delta, 1]^n, \exists i \in [M], D\left(P_{Z_G | \beta} \parallel P_{Z_G | \beta^{(i)}}\right) \leq \varepsilon \quad (41)$$

where we use KL divergence as our ‘‘distance’’ measure. Furthermore, for every  $\varepsilon > 0$ , we define the  $\varepsilon$ -covering number as:

$$M^*(\varepsilon) \triangleq \min\{M \in \mathbb{N} : \exists \varepsilon\text{-covering of } [\delta, 1]^n \text{ with cardinality } M\}. \quad (42)$$

The next lemma distills the upper bound on mutual information via covering numbers presented in (Yang and Barron 1999, Equation (2), p.1571), and specializes it to our setting of (40).

**LEMMA 4 (Covering Number Bound (Wu 2019, Lemma 16.1)).** *Let  $(\alpha_1, \dots, \alpha_n)$  be i.i.d. with distribution  $P_\alpha = \text{unif}([\delta, 1])$ , and recall that the conditional probability distribution of  $Z_G$  given  $(\alpha_1, \dots, \alpha_n)$  is defined by (1) and (3). Then, the mutual information between  $(\alpha_1, \dots, \alpha_n)$  and  $Z_G$  is upper bounded by:*

$$I(\alpha_1, \dots, \alpha_n; Z_G) \leq \inf_{\varepsilon > 0} \varepsilon + \log(M^*(\varepsilon)).$$

Using Lemmata 3 and 4, we can finally prove the upper bound on  $I(\pi, Z)$  in Proposition 2.

*Proof of Proposition 2.* First, notice that  $\pi \rightarrow (\alpha_1, \dots, \alpha_n) \rightarrow Z$  forms a Markov chain, because  $\pi$  is a deterministic function of  $(\alpha_1, \dots, \alpha_n)$  (according to (15)). Hence, by part 2 of Lemma 3, we get:

$$I(\pi; Z) \leq I(\alpha_1, \dots, \alpha_n; Z). \quad (43)$$

Furthermore, notice that  $Z \rightarrow (Z, \mathcal{G}(n, p)) \rightarrow (\alpha_1, \dots, \alpha_n)$  forms a Markov chain, because  $Z$  is a deterministic (projection) function of  $(Z, \mathcal{G}(n, p))$ . Thus, by part 2 of Lemma 3, we also get:

$$\begin{aligned} I(\alpha_1, \dots, \alpha_n; Z) &\leq I(\alpha_1, \dots, \alpha_n; Z, \mathcal{G}(n, p)) \\ &= I(\alpha_1, \dots, \alpha_n; \mathcal{G}(n, p)) + I(\alpha_1, \dots, \alpha_n; Z | \mathcal{G}(n, p)) \\ &= I(\alpha_1, \dots, \alpha_n; Z | \mathcal{G}(n, p)) \end{aligned} \tag{44}$$

where the second equality utilizes part 1 of Lemma 3, and the third equality holds because  $(\alpha_1, \dots, \alpha_n)$  and  $\mathcal{G}(n, p)$  are independent, which implies that  $I(\alpha_1, \dots, \alpha_n; \mathcal{G}(n, p)) = 0$  (see (36)). Combining (43) and (44), we obtain:

$$I(\pi; Z) \leq I(\alpha_1, \dots, \alpha_n; Z | \mathcal{G}(n, p)). \tag{45}$$

So, it suffices to upper bound the conditional mutual information  $I(\alpha_1, \dots, \alpha_n; Z | \mathcal{G}(n, p))$ .

To this end, we condition on any fixed realization of the underlying Erdős-Rényi random graph  $\mathcal{G}(n, p) = G$  (as in the discussion preceding this proof), and proceed to establishing an upper bound on  $I(\alpha_1, \dots, \alpha_n; Z_G)$  by employing Lemma 4. Specifically, we next evaluate the right hand side of the inequality in Lemma 4 above for a judiciously chosen  $\varepsilon$ -covering of  $[\delta, 1]^n$ . Fix any  $q > 0$  (to be chosen later), and quantize the interval  $[\delta, 1]$  using the set of values:

$$\mathcal{Q} \triangleq \left\{ \delta + \frac{(1-\delta)m}{n^q} : m \in [n^q] \right\}$$

which has cardinality  $|\mathcal{Q}| = \lfloor n^q \rfloor \leq n^q$ , and satisfies the condition:

$$\forall t \in [\delta, 1], \quad \min_{s \in \mathcal{Q}} |t - s| \leq \frac{1-\delta}{n^q} \tag{46}$$

where the right hand side can be improved to  $(1-\delta)/(2n^q)$  when  $t$  is not located at the edges of the interval  $[\delta, 1]$ . The next claim shows that  $\mathcal{Q}^n$  is actually an  $\varepsilon$ -covering of  $[\delta, 1]^n$  with  $\varepsilon = O(n^{2-2q})$  (neglecting the dependence of  $\varepsilon$  on  $\delta$  and  $k$ ).

**CLAIM 4 ( $\varepsilon$ -Covering).**  $\mathcal{Q}^n$  is an  $\varepsilon$ -covering of  $[\delta, 1]^n$  with cardinality  $|\mathcal{Q}^n| = (\lfloor n^q \rfloor)^n \leq n^{qn}$  and:

$$\varepsilon = \frac{(1-\delta)^2}{4\delta^2} \left( 2 + \delta + \frac{1}{\delta} \right) \frac{k|G|}{n^{2q}}$$

where  $|G|$  denotes the number of edges in the graph  $G$  with abuse of notation. (Since  $|G| \leq \frac{n(n-1)}{2}$ ,  $\varepsilon = O(n^{2-2q})$ .)

To prove this claim, we will need the following coordinate-wise Lipschitz continuity property.

**CLAIM 5 (Coordinate-wise Lipschitz Continuity).** Consider the map  $F : [\delta, \infty)^2 \rightarrow (0, \infty)$ :

$$\forall x, y \geq \delta, \quad F(x, y) \triangleq \frac{x}{x+y}$$

which is used to define the likelihoods of the BTL model in (1). This map is coordinate-wise Lipschitz continuous:

1. For any fixed  $x \in [\delta, \infty)$ :

$$\forall y_1, y_2 \in [\delta, \infty), \quad |F(x, y_1) - F(x, y_2)| \leq \frac{1}{4\delta} |y_1 - y_2|.$$

2. For any fixed  $y \in [\delta, \infty)$ :

$$\forall x_1, x_2 \in [\delta, \infty), \quad |F(x_1, y) - F(x_2, y)| \leq \frac{1}{4\delta} |x_1 - x_2|.$$

Claim 5 is established in Appendix A.3. We next derive Claim 4 using Claim 5.

*Proof of Claim 4.* The cardinality of  $\mathcal{Q}^n$  follows since it is a product set. So, we focus on verifying the value of  $\varepsilon$ . Fix any parameter vector  $\beta = (\beta_1, \dots, \beta_n) \in [\delta, 1]^n$ . Due to (46), we have that:

$$\forall i \in [n], \exists \gamma_i \in \mathcal{Q}, \quad |\beta_i - \gamma_i| \leq \frac{1 - \delta}{n^q}. \quad (47)$$

Letting  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathcal{Q}^n$ , observe that:

$$\begin{aligned} D(P_{Z_G|\beta} \| P_{Z_G|\gamma}) &= \sum_{\substack{i,j \in [n]: \\ \{i,j\} \in G, i < j}} D(P_{Z(i,j)|\beta,G} \| P_{Z(i,j)|\gamma,G}) \\ &= \sum_{\substack{i,j \in [n]: \\ \{i,j\} \in G, i < j}} \sum_{i=0}^k \binom{k}{i} \left(\frac{\beta_j}{\beta_i + \beta_j}\right)^i \left(\frac{\beta_i}{\beta_i + \beta_j}\right)^{k-i} \log \left( \frac{\left(\frac{\beta_j}{\beta_i + \beta_j}\right)^i \left(\frac{\beta_i}{\beta_i + \beta_j}\right)^{k-i}}{\left(\frac{\gamma_j}{\gamma_i + \gamma_j}\right)^i \left(\frac{\gamma_i}{\gamma_i + \gamma_j}\right)^{k-i}} \right) \\ &= \sum_{\substack{i,j \in [n]: \\ \{i,j\} \in G, i < j}} \log \left( \frac{\left(\frac{\beta_j}{\beta_i + \beta_j}\right)}{\left(\frac{\gamma_j}{\gamma_i + \gamma_j}\right)} \right) \sum_{i=0}^k i \binom{k}{i} \left(\frac{\beta_j}{\beta_i + \beta_j}\right)^i \left(\frac{\beta_i}{\beta_i + \beta_j}\right)^{k-i} \\ &\quad + \log \left( \frac{\left(\frac{\beta_i}{\beta_i + \beta_j}\right)}{\left(\frac{\gamma_i}{\gamma_i + \gamma_j}\right)} \right) \sum_{i=0}^k (k-i) \binom{k}{i} \left(\frac{\beta_j}{\beta_i + \beta_j}\right)^i \left(\frac{\beta_i}{\beta_i + \beta_j}\right)^{k-i} \\ &= k \sum_{\substack{i,j \in [n]: \\ \{i,j\} \in G, i < j}} \left(\frac{\beta_j}{\beta_i + \beta_j}\right) \log \left( \frac{\left(\frac{\beta_j}{\beta_i + \beta_j}\right)}{\left(\frac{\gamma_j}{\gamma_i + \gamma_j}\right)} \right) + \left(\frac{\beta_i}{\beta_i + \beta_j}\right) \log \left( \frac{\left(\frac{\beta_i}{\beta_i + \beta_j}\right)}{\left(\frac{\gamma_i}{\gamma_i + \gamma_j}\right)} \right) \\ &= k \sum_{\substack{i,j \in [n]: \\ \{i,j\} \in G, i < j}} D \left( \frac{\beta_j}{\beta_i + \beta_j} \middle\| \frac{\gamma_j}{\gamma_i + \gamma_j} \right) \\ &\leq k \sum_{\substack{i,j \in [n]: \\ \{i,j\} \in G, i < j}} \left( \frac{\beta_j}{\beta_i + \beta_j} - \frac{\gamma_j}{\gamma_i + \gamma_j} \right)^2 \left( 2 + \frac{\gamma_i}{\gamma_j} + \frac{\gamma_j}{\gamma_i} \right) \end{aligned} \quad (48)$$

$$\begin{aligned}
&\leq k \left(2 + \delta + \frac{1}{\delta}\right) \sum_{\substack{i,j \in [n]: \\ \{i,j\} \in G, i < j}} \left( \frac{\beta_j}{\beta_i + \beta_j} - \frac{\gamma_j}{\gamma_i + \gamma_j} \right)^2 \\
&\leq k \left(2 + \delta + \frac{1}{\delta}\right) \sum_{\substack{i,j \in [n]: \\ \{i,j\} \in G, i < j}} \left( \left| \frac{\beta_j}{\beta_i + \beta_j} - \frac{\beta_j}{\gamma_i + \beta_j} \right| + \left| \frac{\beta_j}{\gamma_i + \beta_j} - \frac{\gamma_j}{\gamma_i + \gamma_j} \right| \right)^2 \\
&\leq \frac{k}{16\delta^2} \left(2 + \delta + \frac{1}{\delta}\right) \sum_{\substack{i,j \in [n]: \\ \{i,j\} \in G, i < j}} (|\beta_i - \gamma_i| + |\beta_j - \gamma_j|)^2 \\
&\leq \frac{(1-\delta)^2}{4\delta^2} \left(2 + \delta + \frac{1}{\delta}\right) \frac{k|G|}{n^{2q}} \\
&= \varepsilon
\end{aligned}$$

where the first equality uses (35), (39), and the fact that the  $Z(i, j)$ 's in (39) are conditionally independent given the skill parameters and the random graph realization  $G$ , and  $P_{Z(i,j)|\beta, G}$  denotes the conditional probability distribution of  $Z(i, j)$  given  $(\alpha_1, \dots, \alpha_n) = \beta$  and  $\mathcal{G}(n, p) = G$ , the second equality holds because each  $kZ(i, j) = \sum_{m=1}^k Z_m(i, j)$  has binomial distribution given  $\{i, j\} \in G$  (as indicated earlier, where the likelihoods of the  $Z_m(i, j)$ 's are defined via (1)), the fourth equality follows from applying the formula for expected values of binomial random variables, the fifth equality uses the binary KL divergence function, which is defined as  $(0, 1) \ni (a, b) \mapsto D(a||b) \triangleq a \log\left(\frac{a}{b}\right) + (1-a) \log\left(\frac{1-a}{1-b}\right)$ , the sixth inequality follows from a simple upper bound on KL divergence in terms of  $\chi^2$ -divergence (Su 1995) (alternatively see, e.g., (Makur and Zheng 2020, Equation (4), Lemma 3) and the references therein for a detailed exposition):

$$\forall a, b \in (0, 1), \quad D(a||b) \leq \underbrace{\frac{(a-b)^2}{b} + \frac{(a-b)^2}{1-b}}_{\text{binary } \chi^2\text{-divergence}} = \frac{(a-b)^2}{b(1-b)},$$

the seventh inequality holds because the function  $g : [\delta, \frac{1}{\delta}] \rightarrow \mathbb{R}$ ,  $g(t) = t + \frac{1}{t}$  is maximized at  $g(\delta) = g(\frac{1}{\delta}) = \delta + \frac{1}{\delta}$  (where  $\frac{\gamma_i}{\gamma_j} \in [\delta, \frac{1}{\delta}]$ ), the eighth inequality follows from the triangle inequality, the ninth inequality follows from Claim 5, and the tenth inequality follows from (47). This establishes the claim.  $\square$

Finally, proceeding with the proof of Proposition 2, using Lemma 4 and Claim 4, we get:

$$\begin{aligned}
I(\alpha_1, \dots, \alpha_n; Z_G) &\leq \varepsilon + \log(M^*(\varepsilon)) \\
&\leq \frac{(1-\delta)^2}{4\delta^2} \left(2 + \delta + \frac{1}{\delta}\right) \frac{k|G|}{n^{2q}} + \log(|\mathcal{Q}^n|) \\
&\leq \frac{(1-\delta)^2}{4\delta^2} \left(2 + \delta + \frac{1}{\delta}\right) \frac{k|G|}{n^{2q}} + qn \log(n).
\end{aligned}$$

Then, by taking expectations on both sides of this inequality with respect to the law of  $\mathcal{G}(n, p)$ , we obtain, using (38), (40), and (45), that:

$$I(\pi; Z) \leq \frac{(1-\delta)^2}{8\delta^2} \left(2 + \delta + \frac{1}{\delta}\right) \frac{kpn(n-1)}{n^{2q}} + qn \log(n)$$



$$\leq qn \log(n) + \frac{(1-\delta)^2}{8\delta^2} \left(2 + \delta + \frac{1}{\delta}\right) kpn^{2-2q}.$$

Neglecting  $\delta$ ,  $k$ , and  $p$ , it is clear that setting  $q \geq \frac{1}{2}$  ensures that this upper bound is  $O(n \log(n))$ . As the proofs of Theorems 1 and 2 in Sections 5.4 and 5.5 illustrate, choosing the smallest  $q$  satisfying  $q \geq \frac{1}{2}$  yields the tightest possible minimax lower bound using this approach. Thus, we set  $q = \frac{1}{2}$  in the above bound and get:

$$I(\pi; Z) \leq \frac{1}{2}n \log(n) + \frac{(1-\delta)^2}{8\delta^2} \left(2 + \delta + \frac{1}{\delta}\right) kpn$$

as desired.  $\square$

In the literature, various information inequalities are often used to upper bound mutual information terms like  $I(\alpha_1, \dots, \alpha_n; Z_G)$ , for any realization  $\mathcal{G}(n, p) = G$ , in a simpler manner (cf. Chen et al. 2016, Equation (44)). However, these approaches tend to yield poorer scaling in the upper bound with  $n$  compared to the covering number argument we utilize (in Lemma 4). For example, the convexity of KL divergence immediately yields the bound (see (Chen et al. 2016, Equation (44)) or (Zhang 2006, p.1319)):

$$\begin{aligned} I(\alpha_1, \dots, \alpha_n; Z_G) &\leq \frac{1}{(1-\delta)^{2n}} \int_{[\delta, 1]^n} \int_{[\delta, 1]^n} D(P_{Z_G|\beta} \| P_{Z_G|\gamma}) d\gamma d\beta \\ &\leq \sup_{\beta, \gamma \in [\delta, 1]^n} D(P_{Z_G|\beta} \| P_{Z_G|\gamma}) \\ &= k \sup_{\beta, \gamma \in [\delta, 1]^n} \sum_{\substack{i, j \in [n]: \\ \{i, j\} \in G, i < j}} D\left(\frac{\beta_j}{\beta_i + \beta_j} \middle\| \frac{\gamma_j}{\gamma_i + \gamma_j}\right) \\ &\leq k|G| \max_{a, b \in [\frac{\delta}{1+\delta}, \frac{1}{1+\delta}]} D(a|b) \\ &= k|G| D\left(\frac{1}{1+\delta} \middle\| \frac{\delta}{1+\delta}\right) \end{aligned} \tag{49}$$

where the third equality follows from (48) (and we use the notation  $\beta = (\beta_1, \dots, \beta_n)$  and  $\gamma = (\gamma_1, \dots, \gamma_n)$  as before), the maximization in the fourth inequality is over  $a, b \in [\frac{\delta}{1+\delta}, \frac{1}{1+\delta}]$  because the map  $0 < x \mapsto x/(c+x)$  is monotone increasing for every fixed  $c > 0$ , and the last equality follows from basic properties of binary KL divergence (see, e.g., (Polyanskiy and Wu 2017, p.20)). As before, by taking expectations on both sides of (49) with respect to the law of  $\mathcal{G}(n, p)$ , we obtain, using (38), (40), and (45), that:

$$I(\pi; Z) \leq \frac{kpn(n-1)}{2} D\left(\frac{1}{1+\delta} \middle\| \frac{\delta}{1+\delta}\right). \tag{50}$$

Neglecting  $\delta$ ,  $k$ , and  $p$ , it is clear that  $I(\pi; Z) = O(n^2)$  in (50), but the proof of Proposition 2 gives the sharper estimate  $I(\pi; Z) = O(n \log(n))$ .

## 5.2. Generalized Fano's Method

We next introduce a canonical approach to obtaining minimax lower bounds in non-parametric estimation problems—the so called *Fano's method*, which was introduced in (Ibragimov and Khas'minskii 1977, Khas'minskii 1979) (also see, e.g., (Yu 1997) and (Tsybakov 2009, Section 2.7.1) for modern treatments). Fano's method proceeds by first lower bounding minimax risk by a Bayes risk, where all the prior probability mass is placed over a suitably chosen (and large) finite set of parameters in the (non-parametric or infinite-dimensional) parameter space, then lower bounding this Bayes risk using the probability of error of a multiple hypothesis testing problem, and finally, lower bounding this probability of error using the well-known *Fano's inequality* from information theory (cf. Cover and Thomas 2006, Theorem 2.10.1). In the problem of estimating (15) based on  $Z$ , the parameter space of the minimax risks in Theorems 1 and 2 is the infinite-dimensional family of PDFs  $\mathcal{P}$ . For simplicity and analytical tractability, instead of directly applying Fano's method to this large parameter space  $\mathcal{P}$ , which would involve constructing a prior distribution over some judiciously chosen finite subset of  $\mathcal{P}$ , we first obtain lower bounds on the minimax risks in Theorems 1 and 2 in terms of Bayes risks. In particular, as discussed earlier, we set  $P_\alpha = \text{unif}([\delta, 1]) \in \mathcal{P}$  throughout Section 5 so that  $\alpha_1, \dots, \alpha_n$  are i.i.d.  $P_\alpha = \text{unif}([\delta, 1])$ . Hence,  $\mathbb{P}(\cdot)$  denotes the joint probability law of  $\alpha_1, \dots, \alpha_n$ ,  $\mathcal{G}(n, p)$ , and  $\{Z_m(i, j) : \{i, j\} \in \mathcal{G}(n, p), m \in [k]\}$  with  $P_\alpha = \text{unif}([\delta, 1])$  in the sequel, and  $\mathbb{E}[\cdot]$  denotes the corresponding expectation operator. This yields the following lower bound on the minimax relative  $\ell^q$ -norm risk for any  $q \in [1, +\infty]$ :

$$\inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}} \mathbb{E}_{P_\alpha} \left[ \frac{\|\hat{\pi} - \pi\|_q}{\|\pi\|_q} \right] \geq \inf_{\hat{\pi}} \mathbb{E} \left[ \frac{\|\hat{\pi} - \pi\|_q}{\|\pi\|_q} \right] \quad (51)$$

where the infima are over all (measurable) randomized estimators  $\hat{\pi} \in \mathcal{S}_n$  of the canonically scaled skill parameters  $\pi$  based on the observation matrix  $Z$ , and  $\mathbb{E}_{P_\alpha}[\cdot]$  denotes the expectation operator with respect to general (not necessarily uniform)  $P_\alpha$ . Clearly, letting  $q = +\infty$  and  $q = 1$  yield the minimax problems in Theorems 1 and 2, respectively. Therefore, we can focus on the simpler problem of lower bounding the Bayes risks on the right hand side of (51) for  $q \in \{1, +\infty\}$ .

Unfortunately, while Fano's method is very effective at lower bounding non-parametric minimax risks, it cannot lower bound Bayes risks where the parameter space is not a discrete and finite set, because the classical Fano's inequality only holds for discrete and finite parameter sets (cf. Cover and Thomas 2006, Theorem 2.10.1). To remedy this dearth of Fano-based techniques to lower bound Bayes risks where the parameter space is a continuum, the so called *generalized Fano's method* has been developed in the recent literature (Zhang 2006, Chen et al. 2016, Xu and Raginsky 2017). One of the first results in this line of work was a generalization of Fano's inequality to the *continuum Fano inequality* in (Duchi and Wainwright 2013, Proposition 2), which had useful

consequences for minimax estimation with a specific zero-one valued loss function (Duchi and Wainwright 2013, Section 3). The techniques of Duchi and Wainwright (2013) have been vastly generalized by Chen et al. (2016) and Xu and Raginsky (2017) to obtain lower bounds on Bayes risks in terms of  $f$ -informativity (cf. Csiszár 1972) and conditional mutual information (with auxiliary random variables), respectively. In this paper, we will utilize the key result in (Xu and Raginsky 2017, Theorem 1, Equation (6)). The lemma below presents the result in (Xu and Raginsky 2017, Theorem 1, Equation (6)) specialized to our relative  $\ell^q$ -loss setting.

**LEMMA 5 (Generalized Fano's Method (Xu and Raginsky 2017, Theorem 1)).** *For any  $q \in [1, +\infty]$ , the Bayes risk on the right hand side of (51) is lower bounded by:*

$$\inf_{\hat{\pi}} \mathbb{E} \left[ \frac{\|\hat{\pi} - \pi\|_q}{\|\pi\|_q} \right] \geq \sup_{t>0} t \left( 1 - \frac{I(\pi; Z) + \log(2)}{\log(1/\mathcal{L}_q(t))} \right)$$

where we define the small ball probability  $\mathcal{L}_q(\cdot)$  as (cf. Xu and Raginsky 2017, Equation (2)):

$$\forall t > 0, \quad \mathcal{L}_q(t) \triangleq \sup_{\nu \in \mathcal{S}_n} \mathbb{P} \left( \frac{\|\pi - \nu\|_q}{\|\pi\|_q} \leq t \right), \quad (52)$$

and  $I(\pi; Z)$  denotes the mutual information (defined in (36)) between the canonically scaled skill parameters  $\pi$  (defined in (15)) and the observation matrix  $Z$  (defined in (3)).

We remark that several variants of Lemma 5 exist in the literature, such as (Zhang 2006, Theorem 6.1) and (Chen et al. 2016, Remark 10, Corollary 12(i)). As expounded in (Chen et al. 2016), in order to compute lower bounds such as that in Lemma 5, we need to establish two things:

1. Tight upper bounds on the mutual information  $I(\pi; Z)$ ,
2. Tight upper bounds on the small ball probability  $\mathcal{L}_q(t)$ .

We have already derived an upper bound on  $I(\pi; Z)$  in Proposition 2 using the covering number argument presented in Section 5.1. Next, we prove upper bounds on the small ball probability  $\mathcal{L}_q(t)$  for  $q \in \{1, +\infty\}$ .

### 5.3. Upper Bounds on Small Ball Probability

As noted by both Chen et al. (2016) and Xu and Raginsky (2017), there is no general recipe for obtaining upper bounds on  $\mathcal{L}_q(t)$ . So, we develop our bounds via direct computation. To this end, the ensuing lemma presents an upper bound on the mode of the joint PDF of  $\pi$ , or more precisely, the joint PDF of:

$$\tilde{\pi} \triangleq (\pi(1), \dots, \pi(n-1)) \quad (53)$$

with respect to the Lebesgue measure on  $\mathbb{R}^{n-1}$ , which excludes  $\pi(n)$ , because  $\pi(n) = 1 - \pi(1) - \dots - \pi(n-1)$ .

LEMMA 6 (**Bound on Mode of Joint PDF of  $\tilde{\pi}$** ). *Let the joint PDF of  $\tilde{\pi}$  with respect to the Lebesgue measure on  $\mathbb{R}^{n-1}$  be denoted  $P_{\tilde{\pi}}$ . The mode of  $P_{\tilde{\pi}}$  is upper bounded by:*

$$\operatorname{ess\,sup}_{\tau \in \mathbb{R}^{n-1}} P_{\tilde{\pi}}(\tau) \leq \frac{n^{n-1}}{(1-\delta)^n}$$

where  $\operatorname{ess\,sup}$  denotes the essential supremum.

*Proof.* First, consider the map  $h : [\delta, 1]^n \rightarrow \operatorname{im}(h)$ :

$$\forall \beta = (\beta_1, \dots, \beta_n) \in [\delta, 1]^n, \quad h(\beta) \triangleq \left( \frac{\beta_1}{\sum_{i=1}^n \beta_i}, \dots, \frac{\beta_{n-1}}{\sum_{i=1}^n \beta_i}, \sum_{i=1}^n \beta_i \right)$$

where  $\operatorname{im}(h) \triangleq \{(\tau_1, \dots, \tau_{n-1}, \sigma) \in [\frac{\delta}{n-1+\delta}, \frac{1}{1+\delta(n-1)}]^{n-1} \times [n\delta, n] : \exists \beta_1, \dots, \beta_n \in [\delta, 1] \text{ such that } \sigma = \sum_{j=1}^n \beta_j \text{ and for all } i \in [n-1], \tau_i = \beta_i/\sigma\}$  denotes the range (or image) of  $h$ . Clearly, we have  $h(\alpha_1, \dots, \alpha_n) = (\tilde{\pi}, \alpha_1 + \dots + \alpha_n)$  using (15) and (53). Furthermore,  $h$  is a bijection with inverse function  $h^{-1} : \operatorname{im}(h) \rightarrow [\delta, 1]^n$ :

$$\forall (\tau_1, \dots, \tau_{n-1}, \sigma) \in \operatorname{im}(h), \quad h^{-1}(\tau_1, \dots, \tau_{n-1}, \sigma) = \left( \sigma\tau_1, \dots, \sigma\tau_{n-1}, \sigma \left( 1 - \sum_{i=1}^{n-1} \tau_i \right) \right).$$

By direct evaluation, the *Jacobian matrix* of  $h^{-1}$ , denoted  $\nabla h^{-1} : \operatorname{im}(h) \rightarrow \mathbb{R}^{n \times n}$ , is:

$$\forall (\tau_1, \dots, \tau_{n-1}, \sigma) \in \operatorname{im}(h), \quad [\nabla h^{-1}]_{i,j} = \begin{cases} \sigma \mathbb{1}\{i=j\}, & i, j \in [n-1] \\ -\sigma, & i=n, j \in [n-1] \\ \tau_i, & i \in [n-1], j=n \\ 1 - \sum_{i=1}^{n-1} \tau_i, & i=j=n \end{cases}$$

where  $[\nabla h^{-1}]_{i,j}$  denotes the  $(i, j)$ th entry of the matrix  $\nabla h^{-1}$  for  $i, j \in [n]$ . (Note that  $\nabla h^{-1}$  is also well-defined on the boundary of  $\operatorname{im}(h)$ , because there exists an open set containing  $\operatorname{im}(h)$  such that the first partial derivatives of  $h^{-1}$  exist on this open set.) Now define the successive sub-matrices:

$$\forall r \in \{0, 1, \dots, n-2\}, \quad M_{n-r} \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 & \tau_{r+1} \\ 0 & 1 & \cdots & 0 & \tau_{r+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \tau_{n-1} \\ -1 & -1 & \cdots & -1 & 1 - \sum_{i=1}^{n-1} \tau_i \end{bmatrix} \in \mathbb{R}^{(n-r) \times (n-r)}$$

where  $M_n$  is closely related to  $\nabla h^{-1}$  (as shown below in (54)), and let the transpose of the *Frobenius companion matrix* of the monic polynomial  $q_n(t) = 1 + t + t^2 + \dots + t^n$  be (cf. Horn and Johnson 2013, Definition 3.3.13):

$$C_n \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & -1 & -1 & \cdots & -1 \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

Then, the corresponding *Jacobian determinant* satisfies the recurrence relation:

$$\begin{aligned} \forall(\tau_1, \dots, \tau_{n-1}, \sigma) \in \text{im}(h), \quad \det(\nabla h^{-1}) &= \det \left( \begin{bmatrix} \sigma & 0 & \cdots & 0 & \tau_1 \\ 0 & \sigma & \cdots & 0 & \tau_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma & \tau_{n-1} \\ -\sigma & -\sigma & \cdots & -\sigma & 1 - \sum_{i=1}^{n-1} \tau_i \end{bmatrix} \right) \\ &= \sigma^{n-1} \det(M_n) \end{aligned} \quad (54)$$

$$= \sigma^{n-1} (\det(M_{n-1}) + (-1)^{n+1} \tau_1 \det(C_{n-1})) \quad (55)$$

where (54) follows from the multilinearity of the determinant, and (55) uses the *Laplace (cofactor) expansion* of determinants by minors along the first row (see, e.g., [Horn and Johnson 2013](#), Section 0.3.1). We next compute this Jacobian determinant.

It is easy to calculate  $\det(C_{n-1})$  in (55), because  $q_{n-1}(t)$  is also the characteristic polynomial of its (adjoint) companion matrix  $C_{n-1}$  (cf. [Horn and Johnson 2013](#), Theorem 3.3.14). The  $n-1$  distinct roots of  $q_{n-1}(t)$  are the following  $n$ th roots of unity:

$$\forall r \in [n-1], \quad q_{n-1}(\omega^r) = 0$$

where  $\omega = \exp\left(\frac{2\pi i}{n}\right)$ . (Note that unlike the rest of this paper, in the definition of  $\omega$ , we use  $\mathbf{i}$  and  $\pi$  to represent the imaginary unit  $\mathbf{i} = \sqrt{-1}$  and the mathematical constant  $\pi = 3.14159\dots$ , respectively.) Hence,  $\{\omega^r : r \in [n-1]\}$  are the eigenvalues of  $C_{n-1}$ , and we have:

$$\det(C_{n-1}) = \prod_{r=1}^{n-1} \omega^r = \omega^{n(n-1)/2} = (-1)^{n-1}$$

since the determinant is the product of the eigenvalues (see, e.g., [Horn and Johnson 2013](#), Section 1.2). Combining this with (55), we get:

$$\begin{aligned} \forall(\tau_1, \dots, \tau_{n-1}, \sigma) \in \text{im}(h), \quad \det(\nabla h^{-1}) &= \sigma^{n-1} (\det(M_{n-1}) + (-1)^{n+1} \tau_1 (-1)^{n-1}) \\ &= \sigma^{n-1} \underbrace{(\det(M_{n-1}) + \tau_1)}_{= \det(M_n)} \\ &= \sigma^{n-1} \left( \det(M_2) + \sum_{j=1}^{n-2} \tau_j \right) \\ &= \sigma^{n-1} \left( 1 - \sum_{j=1}^{n-1} \tau_j + \tau_{n-1} + \sum_{j=1}^{n-2} \tau_j \right) \\ &= \sigma^{n-1} \end{aligned} \quad (56)$$

where the third equality follows from unwinding the recursion in the second line.

Now observe that the joint PDF of  $(\alpha_1, \dots, \alpha_n)$  (with respect to the Lebesgue measure on  $\mathbb{R}^n$ ) is given by:

$$\forall \beta \in \mathbb{R}^n, P_{\alpha_1, \dots, \alpha_n}(\beta) = \frac{1}{(1-\delta)^n} \mathbb{1}\{\beta \in [\delta, 1]^n\}$$

since  $\alpha_1, \dots, \alpha_n$  are i.i.d.  $P_\alpha = \text{unif}([\delta, 1])$ . As a consequence, the joint PDF of  $h(\alpha_1, \dots, \alpha_n) = (\tilde{\pi}, \alpha_1 + \dots + \alpha_n)$  (with respect to the Lebesgue measure on  $\mathbb{R}^n$ ) is given by the *change-of-variables formula*:

$$\begin{aligned} P_{\tilde{\pi}, \alpha_1 + \dots + \alpha_n}(\tau_1, \dots, \tau_{n-1}, \sigma) &= P_{\alpha_1, \dots, \alpha_n}(h^{-1}(\tau_1, \dots, \tau_{n-1}, \sigma)) |\det(\nabla h^{-1})| \\ &= \frac{\sigma^{n-1}}{(1-\delta)^n} \mathbb{1}\{(\tau_1, \dots, \tau_{n-1}, \sigma) \in \text{im}(h)\} \end{aligned} \quad (57)$$

for all  $(\tau_1, \dots, \tau_{n-1}, \sigma) \in \mathbb{R}^n$ , where we utilize our earlier computation of the Jacobian determinant in (56). Although we only seek to bound the joint PDF of  $\tilde{\pi}$ , the joint PDF in (57) includes an additional random variable  $\alpha_1 + \dots + \alpha_n$  as an artifact of our calculation approach (which requires an invertible map  $h$  with a well-defined and invertible Jacobian matrix  $\nabla h$ ).

So, in the final step of this proof, we marginalize the joint PDF in (57) and then bound the desired joint PDF of  $\tilde{\pi}$  (with respect to the Lebesgue measure on  $\mathbb{R}^{n-1}$ ):

$$\begin{aligned} \forall \tau \in \mathbb{R}^{n-1}, P_{\tilde{\pi}}(\tau) &= \mathbb{1}\{\tau \in \tilde{\mathcal{S}}_n\} \int_{[n\delta, n]} P_{\tilde{\pi}, \alpha_1 + \dots + \alpha_n}(\tau, \sigma) d\sigma \\ &= \frac{\mathbb{1}\{\tau \in \tilde{\mathcal{S}}_n\}}{(1-\delta)^n} \int_{[n\delta, n]} \sigma^{n-1} \mathbb{1}\{(\tau, \sigma) \in \text{im}(h)\} d\sigma \\ &\leq \frac{\mathbb{1}\{\tau \in \tilde{\mathcal{S}}_n\}}{(1-\delta)^n} \int_{[n\delta, n]} \sigma^{n-1} d\sigma \\ &= \frac{n^{n-1}(1-\delta^n)}{(1-\delta)^n} \mathbb{1}\{\tau \in \tilde{\mathcal{S}}_n\} \\ &\leq \frac{n^{n-1}}{(1-\delta)^n} \mathbb{1}\{\tau \in \tilde{\mathcal{S}}_n\} \end{aligned}$$

where  $\tilde{\mathcal{S}}_n \triangleq \{(\tau_1, \dots, \tau_{n-1}) \in [\frac{\delta}{n-1+\delta}, \frac{1}{1+\delta(n-1)}]^{n-1} : \exists \beta_1, \dots, \beta_n \in [\delta, 1] \text{ such that for all } i \in [n-1], \tau_i = \beta_i / (\sum_{j=1}^n \beta_j)\}$ . Taking the (essential) supremum over all  $\tau \in \mathbb{R}^{n-1}$  in the above bound completes the proof.  $\square$

We now use Lemma 6 to upper bound the small ball probabilities  $\mathcal{L}_q(t)$  for  $q \in \{1, +\infty\}$  in the lemmata below.

**LEMMA 7 (Upper Bound on Small Ball Probability for  $q = \infty$ ).** *For every  $t > 0$ , we have:*

$$\mathcal{L}_\infty(t) \leq \left( \frac{2}{\delta(1-\delta)} \right)^n t^{n-1}.$$

*Proof.* Starting with (52), observe that:

$$\begin{aligned}
 \forall t > 0, \mathcal{L}_\infty(t) &= \sup_{\nu=(\nu_1, \dots, \nu_n) \in \mathcal{S}_n} \mathbb{P}(\|\pi - \nu\|_\infty \leq t \|\pi\|_\infty) \\
 &\leq \sup_{\nu=(\nu_1, \dots, \nu_n) \in \mathcal{S}_n} \mathbb{P}\left(\|\pi - \nu\|_\infty \leq \frac{t}{\delta n}\right) \\
 &\leq \sup_{\nu=(\nu_1, \dots, \nu_n) \in \mathcal{S}_n} \mathbb{P}\left(\|\tilde{\pi} - (\nu_1, \dots, \nu_{n-1})\|_\infty \leq \frac{t}{\delta n}\right) \\
 &= \sup_{\nu=(\nu_1, \dots, \nu_n) \in \mathcal{S}_n} \int_{\mathbb{R}^{n-1}} P_{\tilde{\pi}}(\tau) \mathbb{1}\left\{\|\tau - (\nu_1, \dots, \nu_{n-1})\|_\infty \leq \frac{t}{\delta n}\right\} d\tau \\
 &\leq \frac{n^{n-1}}{(1-\delta)^n} \sup_{\nu=(\nu_1, \dots, \nu_n) \in \mathcal{S}_n} \underbrace{\int_{\mathbb{R}^{n-1}} \mathbb{1}\left\{\|\tau - (\nu_1, \dots, \nu_{n-1})\|_\infty \leq \frac{t}{\delta n}\right\} d\tau}_{\text{volume of } \ell^\infty\text{-ball with radius } t/(\delta n)} \\
 &= \frac{n^{n-1}}{(1-\delta)^n} \left(\frac{2t}{\delta n}\right)^{n-1} \\
 &\leq \left(\frac{2}{\delta(1-\delta)}\right)^n t^{n-1}
 \end{aligned}$$

where the second inequality uses the bound:

$$\|\pi\|_\infty = \frac{\max_{i \in [n]} \alpha_i}{\sum_{i=1}^n \alpha_i} \leq \frac{1}{\delta n}$$

which follows from (15) and the fact that  $\alpha_1, \dots, \alpha_n \in [\delta, 1]$ , the third inequality uses (53) and the fact that  $\|\tilde{\pi} - (\nu_1, \dots, \nu_{n-1})\|_\infty \leq \|\pi - \nu\|_\infty$ , the fifth inequality follows from Lemma 6, the sixth equality uses the well-known volume of the  $\ell^\infty$ -ball (or hypercube) with radius  $t/(\delta n)$ , and the seventh inequality follows from the fact that  $2/\delta \geq 1$ . This completes the proof.  $\square$

**LEMMA 8 (Upper Bound on Small Ball Probability for  $q = 1$ ).** *For every  $t > 0$ , we have:*

$$\mathcal{L}_1(t) \leq \frac{1}{5\sqrt{n}} \left(\frac{2e}{1-\delta}\right)^n t^{n-1}.$$

Lemma 8 is derived in Appendix A.4; the proof is similar to that of Lemma 7. Next, we provide proofs of Theorems 1 and 2 using Proposition 2, Lemmata 1, 5, 7, and 8, and the result in (Chen et al. 2019, Theorem 5.2).

#### 5.4. Proof of Theorem 1

*Proof of Theorem 1.* We first prove the minimax upper bound. The inequality:

$$\inf_{\tilde{\pi}} \sup_{P_\alpha \in \mathcal{P}} \mathbb{E}_{P_\alpha} \left[ \frac{\|\hat{\pi} - \pi\|_\infty}{\|\pi\|_\infty} \right] \leq \sup_{P_\alpha \in \mathcal{P}} \mathbb{E}_{P_\alpha} \left[ \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \right]$$

holds trivially, because  $\hat{\pi}_* \in \mathcal{S}_n$  in (7) is an estimator for  $\pi$  based on  $Z$ . To prove an upper bound on the extremal Bayes risk on the right hand side of this inequality, we define the event in Lemma 1 as:

$$A \triangleq \left\{ \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \leq \frac{c_4}{\delta} \sqrt{\frac{\log(n)}{npk}} \right\}$$

where  $c_4 > 0$  is the universal constant from Lemma 1. Then, (24) states that for any PDF  $P_\alpha \in \mathcal{P}$  and for all sufficiently large  $n \in \mathbb{N}$ :

$$\mathbb{P}_{P_\alpha}(A) \geq 1 - \frac{c_5}{n^5} \quad (58)$$

where  $c_5 > 0$  is another universal constant from Lemma 1, and  $\mathbb{P}_{P_\alpha}(\cdot)$  denotes the probability measure with respect to general (not necessarily uniform)  $P_\alpha$ . Hence, for every PDF  $P_\alpha \in \mathcal{P}$  and for all sufficiently large  $n$ , we have:

$$\begin{aligned} \mathbb{E}_{P_\alpha} \left[ \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \right] &= \mathbb{E}_{P_\alpha} \left[ \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \middle| A \right] \mathbb{P}_{P_\alpha}(A) + \mathbb{E}_{P_\alpha} \left[ \frac{\|\hat{\pi}_* - \pi\|_\infty}{\|\pi\|_\infty} \middle| A^c \right] \mathbb{P}_{P_\alpha}(A^c) \\ &\leq \frac{c_4}{\delta} \sqrt{\frac{\log(n)}{npk}} + \frac{c_5}{n^4} \\ &\leq \frac{2c_4}{\delta} \sqrt{\frac{\log(n)}{npk}} \end{aligned} \quad (59)$$

where the first equality uses the law of total expectation, the second inequality follows from (58) and the facts that  $\|\pi\|_\infty \geq 1/n$  and  $\|\hat{\pi}_* - \pi\|_\infty \leq 1$ , and the third inequality (59) holds for all sufficiently large  $n$  because  $k = \Theta(1)$ . Letting  $c_{15} = 2c_4/(\delta\sqrt{pk})$  and substituting it into (59), and then taking the supremum in (59) over all PDFs  $P_\alpha \in \mathcal{P}$  yields the desired upper bound in the theorem statement.

We next prove the information theoretic lower bound. Fix any  $\varepsilon > 0$ , and consider any sufficiently large  $n \geq 2$  such that:

$$n \geq \max \left\{ 2 + \frac{1}{\varepsilon}, \left( \frac{2}{\delta(1-\delta)} \right)^{4/\varepsilon}, \exp \left( \frac{(1-\delta)^2(2+\delta+\frac{1}{\delta})kp + 4\log(2)\delta^2}{\delta^2\varepsilon} \right) \right\}. \quad (60)$$

Then, observe that:

$$\begin{aligned} \inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}} \mathbb{E}_{P_\alpha} \left[ \frac{\|\hat{\pi} - \pi\|_\infty}{\|\pi\|_\infty} \right] &\geq \sup_{t>0} t \left( 1 - \frac{I(\pi; Z) + \log(2)}{\log(1/\mathcal{L}_\infty(t))} \right) \\ &\geq \sup_{t>0} t \left( 1 - \frac{\frac{1}{2}n \log(n) + c(\delta, p, k)n + \log(2)}{\log(1/\mathcal{L}_\infty(t))} \right) \\ &\geq \sup_{t>0} t \left( 1 - \frac{\frac{1}{2}n \log(n) + c(\delta, p, k)n + \log(2)}{(n-1) \log(1/t) - \log(2/(\delta(1-\delta)))n} \right) \\ &= \sup_{t>0} t \left( 1 - \frac{1 + \frac{2c(\delta, p, k)}{\log(n)} + \frac{\log(4)}{n \log(n)}}{\frac{2(n-1) \log(1/t)}{n \log(n)} - \frac{2 \log(2/(\delta(1-\delta)))}{\log(n)}} \right) \\ &\geq \frac{1}{n^{\frac{1}{2}+\varepsilon}} \left( 1 - \frac{1 + \frac{2c(\delta, p, k)}{\log(n)} + \frac{\log(4)}{n \log(n)}}{(1+2\varepsilon) \left( 1 - \frac{1}{n} \right) - \frac{2 \log(2/(\delta(1-\delta)))}{\log(n)}} \right) \\ &\geq \frac{1}{n^{\frac{1}{2}+\varepsilon}} \left( 1 - \frac{1 + \frac{\varepsilon}{4}}{1 + \frac{\varepsilon}{2}} \right) \\ &= \frac{1}{n^{\frac{1}{2}+\varepsilon}} \left( \frac{\varepsilon}{4 + 2\varepsilon} \right) \end{aligned} \quad (61)$$



where the first inequality follows from (51) and Lemma 5, the second inequality follows from Proposition 2 and we let:

$$c(\delta, p, k) = \frac{(1-\delta)^2}{8\delta^2} \left(2 + \delta + \frac{1}{\delta}\right) kp \quad (62)$$

for clarity, the third inequality holds due to Lemma 7, the fifth inequality follows from setting  $t = n^{-(1/2)-\varepsilon}$ , and the sixth inequality follows from (60), which implies the following bounds:

$$n \geq \exp\left(\frac{(1-\delta)^2(2 + \delta + \frac{1}{\delta})kp + 4\log(2)\delta^2}{\delta^2\varepsilon}\right) \Rightarrow \frac{2c(\delta, p, k)}{\log(n)} + \frac{\log(4)}{n\log(n)} \leq \frac{\varepsilon}{4}, \quad (63)$$

$$n \geq 2 + \frac{1}{\varepsilon} \Leftrightarrow 1 - \frac{1}{n} \geq \frac{1 + \varepsilon}{1 + 2\varepsilon}, \quad (64)$$

$$n \geq \left(\frac{2}{\delta(1-\delta)}\right)^{4/\varepsilon} \Leftrightarrow \frac{2\log\left(\frac{2}{\delta(1-\delta)}\right)}{\log(n)} \leq \frac{\varepsilon}{2}.$$

Now, let us define the constant:

$$c_{16} = \max\left\{4\log\left(\frac{2}{\delta(1-\delta)}\right), \frac{(1-\delta)^2(2 + \delta + \frac{1}{\delta})kp + 4\log(2)\delta^2}{\delta^2}\right\}$$

and set  $\varepsilon = c_{16}/\log(n)$ . It is straightforward to verify that (60) is satisfied for this choice of  $\varepsilon$  for all sufficiently large  $n$ . Moreover, since  $\varepsilon \leq 1$  for all sufficiently large  $n$ , we see that (61) can be recast as:

$$\inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}} \mathbb{E}_{P_\alpha} \left[ \frac{\|\hat{\pi} - \pi\|_\infty}{\|\pi\|_\infty} \right] \geq \frac{c_{16}}{6\log(n) n^{\frac{1}{2} + \frac{c_{16}}{\log(n)}}} = \left(\frac{c_{16}}{6\exp(c_{16})}\right) \frac{1}{\log(n)\sqrt{n}}$$

for all sufficiently large  $n$ . Finally, letting  $c_{14} = c_{16}/(6\exp(c_{16}))$  yields the minimax lower bound in the theorem statement. This completes the proof.  $\square$

## 5.5. Proof of Theorem 2

Although the upper bound in Theorem 2 can be established using Lemma 1, we can remove an extra  $\sqrt{\log(n)}$  factor by utilizing (Chen et al. 2019, Theorem 5.2) (also see (Negahban et al. 2017, Theorem 2)). So, we present this result in the lemma below.

**LEMMA 9 (Relative  $\ell^2$ -Loss Bound (Chen et al. 2019, Theorem 5.2)).** *Suppose that  $\delta = \Theta(1)$  and  $p \geq c_{19} \log(n)/n$  for some sufficiently large constant  $c_{19} > 0$  (which may depend on  $\delta$ ). Then, there exists a constant  $c_{20} > 0$  (which may depend on  $\delta$ ) and a (universal) constant  $c_{21} > 0$  such that for all sufficiently large  $n \in \mathbb{N}$ , we have:*

$$\mathbb{P}\left(\frac{\|\hat{\pi}_* - \pi\|_2}{\|\pi\|_2} \leq \frac{c_{20}}{\sqrt{npk}} \mid \alpha_1, \dots, \alpha_n\right) \geq 1 - \frac{c_{21}}{n^5}$$

where the probability is computed with respect to the conditional distribution of the observation matrix  $Z$  and the random graph  $\mathcal{G}(n, p)$  given any realizations of the skill parameters  $\alpha_1, \dots, \alpha_n$ , and the estimator  $\hat{\pi}_* \in \mathcal{S}_n$  is defined in (7).

This lemma is an analog of Lemma 1, but for  $\ell^2$ -norm instead of  $\ell^\infty$ -norm. As remarked after Lemma 1, in contrast to this work, the conditioning on  $\alpha_1, \dots, \alpha_n$  in Lemma 9 reflects the non-Bayesian scenario considered in (Chen et al. 2019) (where  $\alpha_1, \dots, \alpha_n$  are deterministic). We next derive Theorem 2.

*Proof of Theorem 2.* The proof strategy is similar to the proof of Theorem 1, but we present the details here since they differ. We first prove the minimax upper bound. As before, the inequality:

$$\inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}} \mathbb{E}_{P_\alpha} [\|\hat{\pi} - \pi\|_1] \leq \sup_{P_\alpha \in \mathcal{P}} \mathbb{E}_{P_\alpha} [\|\hat{\pi}_* - \pi\|_1]$$

holds trivially. To prove an upper bound on the extremal Bayes risk on the right hand side of this inequality, we define the event in Lemma 9 as:

$$A \triangleq \left\{ \frac{\|\hat{\pi}_* - \pi\|_2}{\|\pi\|_2} \leq \frac{c_{20}}{\sqrt{npk}} \right\}.$$

Then, after taking expectations in Lemma 9 with respect to the law of  $\alpha_1, \dots, \alpha_n$ , we get that for any PDF  $P_\alpha \in \mathcal{P}$  and for all sufficiently large  $n \in \mathbb{N}$ :

$$\mathbb{P}_{P_\alpha}(A) \geq 1 - \frac{c_{21}}{n^5}. \quad (65)$$

Hence, for every PDF  $P_\alpha \in \mathcal{P}$  and for all sufficiently large  $n$ , we have:

$$\begin{aligned} \mathbb{E}_{P_\alpha} [\|\hat{\pi}_* - \pi\|_1] &= \mathbb{E}_{P_\alpha} [\|\hat{\pi}_* - \pi\|_1 | A] \mathbb{P}_{P_\alpha}(A) + \mathbb{E}_{P_\alpha} [\|\hat{\pi}_* - \pi\|_1 | A^c] \mathbb{P}_{P_\alpha}(A^c) \\ &\leq \frac{c_{20}}{\delta \sqrt{npk}} + \frac{2c_{21}}{n^5} \\ &\leq \frac{2c_{20}}{\delta \sqrt{npk}} \end{aligned} \quad (66)$$

where the first equality uses the law of total expectation, the second inequality follows from (65) and the facts that  $\|\hat{\pi}_* - \pi\|_1 \leq \|\hat{\pi}_*\|_1 + \|\pi\|_1 = 2$  (via the triangle inequality), and conditioned on  $A$ , we get:

$$\|\hat{\pi}_* - \pi\|_1 \leq \sqrt{n} \|\hat{\pi}_* - \pi\|_2 \leq \frac{c_{20}}{\sqrt{pk}} \|\pi\|_2 \leq \frac{c_{20}}{\delta \sqrt{npk}}$$

which, in turn, uses the equivalence of  $\ell^1$  and  $\ell^2$ -norms (via the Cauchy-Schwarz-Bunyakovsky inequality) and the simple bound  $\|\pi\|_2 \leq 1/(\delta\sqrt{n})$  (due to (15) and  $\alpha_1, \dots, \alpha_n \in [\delta, 1]$ ), and the third inequality (66) holds for all sufficiently large  $n$  because  $k = \Theta(1)$ . Letting  $c_{18} = 2c_{20}/(\delta\sqrt{pk})$  and substituting it into (66), and then taking the supremum in (66) over all PDFs  $P_\alpha \in \mathcal{P}$  yields the desired upper bound in the theorem statement.

We next prove the information theoretic lower bound. As before, fix any  $\varepsilon > 0$ , and consider any sufficiently large  $n \geq 2$  such that:

$$n \geq \max \left\{ 2 + \frac{1}{\varepsilon}, \left( \frac{2e}{1-\delta} \right)^{4/\varepsilon}, \exp \left( \frac{(1-\delta)^2 (2 + \delta + \frac{1}{\delta}) kp + 4 \log(2) \delta^2}{\delta^{2\varepsilon}} \right) \right\}. \quad (67)$$

Then, observe that:

$$\begin{aligned}
\inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}} \mathbb{E}_{P_\alpha} [\|\hat{\pi} - \pi\|_1] &\geq \sup_{t>0} t \left( 1 - \frac{I(\pi; Z) + \log(2)}{\log(1/\mathcal{L}_1(t))} \right) \\
&\geq \sup_{t>0} t \left( 1 - \frac{\frac{1}{2}n \log(n) + c(\delta, p, k)n + \log(2)}{\log(1/\mathcal{L}_1(t))} \right) \\
&\geq \sup_{t>0} t \left( 1 - \frac{\frac{1}{2}n \log(n) + c(\delta, p, k)n + \log(2)}{(n-1) \log(1/t) - \log(2e/(1-\delta))n} \right) \\
&= \sup_{t>0} t \left( 1 - \frac{1 + \frac{2c(\delta, p, k)}{\log(n)} + \frac{\log(4)}{n \log(n)}}{\frac{2(n-1) \log(1/t)}{n \log(n)} - \frac{2 \log(2e/(1-\delta))}{\log(n)}} \right) \\
&\geq \frac{1}{n^{\frac{1}{2}+\varepsilon}} \left( 1 - \frac{1 + \frac{2c(\delta, p, k)}{\log(n)} + \frac{\log(4)}{n \log(n)}}{(1+2\varepsilon)(1-\frac{1}{n}) - \frac{2 \log(2e/(1-\delta))}{\log(n)}} \right) \\
&\geq \frac{1}{n^{\frac{1}{2}+\varepsilon}} \left( 1 - \frac{1 + \frac{\varepsilon}{4}}{1 + \frac{\varepsilon}{2}} \right) \\
&= \frac{1}{n^{\frac{1}{2}+\varepsilon}} \left( \frac{\varepsilon}{4 + 2\varepsilon} \right) \tag{68}
\end{aligned}$$

where the first inequality follows from (51), Lemma 5, and the fact that  $\|\pi\|_1 = 1$ , the second inequality follows from Proposition 2 with  $c(\delta, p, k)$  as defined in (62), the third inequality holds due to the following consequence of Lemma 8:

$$\forall t > 0, \mathcal{L}_1(t) \leq \left( \frac{2e}{1-\delta} \right)^n t^{n-1},$$

the fifth inequality follows from setting  $t = n^{-(1/2)-\varepsilon}$ , and the sixth inequality follows from (67), which implies the bounds (63), (64), and:

$$n \geq \left( \frac{2e}{1-\delta} \right)^{4/\varepsilon} \Leftrightarrow \frac{2 \log\left(\frac{2e}{1-\delta}\right)}{\log(n)} \leq \frac{\varepsilon}{2}.$$

Now, let us define the constant:

$$c_{22} = \max \left\{ 4 \log\left(\frac{2e}{1-\delta}\right), \frac{(1-\delta)^2 \left(2 + \delta + \frac{1}{\delta}\right) kp + 4 \log(2) \delta^2}{\delta^2} \right\}$$

and set  $\varepsilon = c_{22}/\log(n)$ . As mentioned earlier, it is straightforward to verify that (67) is satisfied for this choice of  $\varepsilon$  for all sufficiently large  $n$ . Finally, as before, we can rewrite (68) as:

$$\inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}} \mathbb{E}_{P_\alpha} [\|\hat{\pi} - \pi\|_1] \geq \left( \frac{c_{22}}{6 \exp(c_{22})} \right) \frac{1}{\log(n) \sqrt{n}}$$

for all sufficiently large  $n$ . Letting  $c_{17} = c_{22}/(6 \exp(c_{22}))$  yields the minimax lower bound in the theorem statement. This proves the theorem.  $\square$

## 6. Experiments

In this section, we apply our method to several real-world datasets to exhibit its utility. Specifically, Algorithm 1 produces estimates of skill distributions. In order to compare skill distributions across different scenarios as well as capture their essence, it is desirable to compute a single *score* that holistically measures the variation of levels of skill in a tournament. We propose such a score in Section 6.1, explain our algorithmic choices in Section 6.2, and then portray our numerical experiments on Cricket World Cups, Soccer World Cups, European soccer leagues, and US mutual funds in Sections 6.3, 6.4, 6.5, and 6.6, respectively.

### 6.1. Overall Skill Score

Intuitively, a Dirac delta measure (i.e., all skill levels are equal) represents a setting where all game outcomes are completely random; there is no role of skill. On the other hand, the uniform PDF  $\text{unif}([0, 1])$  (assuming  $\delta$  is very small) typifies a setting where players are endowed with a broad variety of skill levels. We refer readers to (Getty et al. 2018) for a related discussion. Propelled by this intuition, any “distance” between  $P_\alpha$  and  $\text{unif}([0, 1])$  that is maximized when  $P_\alpha$  is a Dirac delta measure serves as a valid score, which is larger when luck plays a greater role in determining the outcomes of games. Therefore, we propose to use the negative *differential entropy* of  $P_\alpha$  as an overall score to measure skill in a tournament (Cover and Thomas 2006, Polyanskiy and Wu 2017):

$$-h(P_\alpha) \triangleq \int_{\mathbb{R}} P_\alpha(t) \log(P_\alpha(t)) dt \in [0, +\infty]. \quad (69)$$

This is a well-defined quantity (for all PDFs  $P_\alpha$ ) that is equal to the KL divergence between  $P_\alpha$  and  $\text{unif}([0, 1])$ :  $-h(P_\alpha) = D(P_\alpha || \text{unif}([0, 1]))$  (cf. (35) in Section 5.1). Moreover,  $-h(P_\alpha) = 0$  when  $P_\alpha = \text{unif}([0, 1])$ , and  $-h(P_\alpha) = +\infty$  when  $P_\alpha$  is a Dirac delta measure. To estimate  $-h(P_\alpha)$  from data, we will use the simple *resubstitution estimator* based on  $\hat{\mathcal{P}}^*$  and  $\hat{\alpha}_1, \dots, \hat{\alpha}_n$  (Ahmad and Lin 1976, Beirlant et al. 1997):

$$-\hat{H}_Z \triangleq \frac{1}{n} \sum_{i=1}^n \log(\hat{\mathcal{P}}^*(\hat{\alpha}_i)) \quad (70)$$

where  $\hat{\mathcal{P}}^*$  and  $\hat{\alpha}_1, \dots, \hat{\alpha}_n$  are the estimators for  $P_\alpha$  and  $\alpha_1, \dots, \alpha_n$  based on the data  $Z$  in (9) and (8), respectively. (Note that using other entropy estimation methods demonstrate the same trends in the sequel.)

### 6.2. Algorithmic Choices and Data Preprocessing

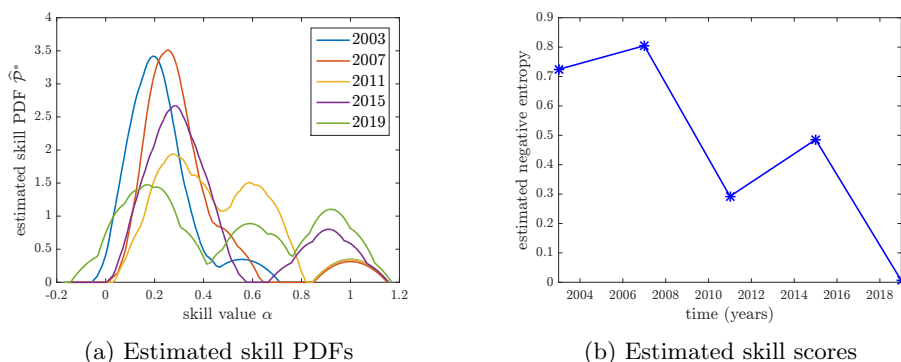
In all our simulations, we assume that  $\eta = 1$ , use the Epanechnikov kernel  $K_E$  defined in (12), and set the bandwidth to  $h = 0.3n^{-1/4}$ ; indeed,  $h$  is typically chosen using ad hoc data-driven techniques in practice (Tsybakov 2009, Section 1.4).

The data is available in the form of wins, losses, and draws in tournaments. For simplicity, we ignore draws and only utilize wins and losses (as in the assumed BTL model). To allow for regularization in the small data regime, we apply Laplace smoothing so that between any pair of players, each observed game is counted as 20 games, and 1 additional win is added for each player; this effectively means that  $p = 1$ .

We remark that the constant 0.3 to define  $h$  and the level of smoothing mentioned above are chosen to generate “smooth” PDFs in Figures 1, 2, 3, and 4. Moreover, the qualitative results and trends in the sequel remain the same for a range of values around the constant 0.3 and the chosen level of smoothing.

### 6.3. Cricket World Cups

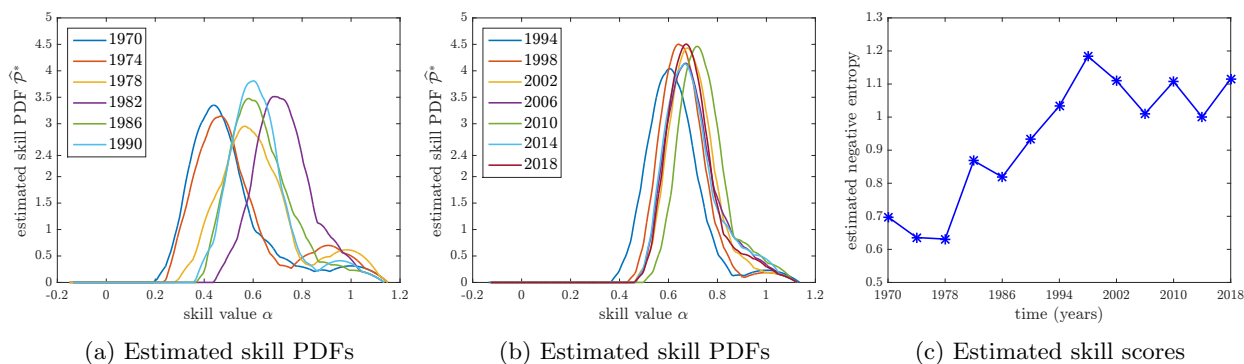
We utilize publicly available data from Wikipedia for international (ICC) Cricket World Cups held in 2003, 2007, 2011, 2015, and 2019. Each World Cup has between  $n = 10$  to  $n = 16$  teams, with each pair of teams playing 0, 1, or (rarely) 2 matches against one another. We learn the skill distributions for each World Cup separately as portrayed in Figure 1a. The corresponding negative entropies are reported in Figure 1b. As can be seen, there is a clear decrease in negative entropy from 2003 to 2019, reaching close to 0 in 2019. This elegantly quantifies sports intuition that the skill levels of Australia and India dominated those of other teams in 2003, and the winners were easy to predict. In contrast, the 2019 World Cup had a far greater variation of skill levels, which meant that many teams were potential contenders for the championship. This led to the 2019 World Cup being perceived as more “exciting” by fans (Smyth et al. 2019, Bull 2019).



**Figure 1** Plots 1a and 1b illustrate the *estimated skill PDFs* and corresponding *estimated negative differential entropies*, respectively, of ICC Cricket World Cups from 2003 to 2019.

## 6.4. Soccer World Cups

Again, we use publicly available data from Wikipedia for FIFA Soccer World Cups in 1970, 1974, 1978,  $\dots$ , 2010, 2014, and 2018. Each World Cup has between  $n = 16$  to  $n = 32$  teams, with each pair of teams playing 0, 1, or (rarely) 2 matches. Figures 2a, 2b, and 2c depict the skill distributions and associated negative entropies of Soccer World Cups over the years. It is evident that the negative entropies have increased from 1978 to 2002, and then plateaued (i.e., remained roughly constant and away from 0) from 2002 onwards. This suggests that the unpredictability of game outcomes in Soccer World Cups has increased over the years and then stabilized—very consistent with fan experience that Soccer World Cups have evolved over the last half-century to become more competitive with far tighter matches (Scott and Kirk 2018).

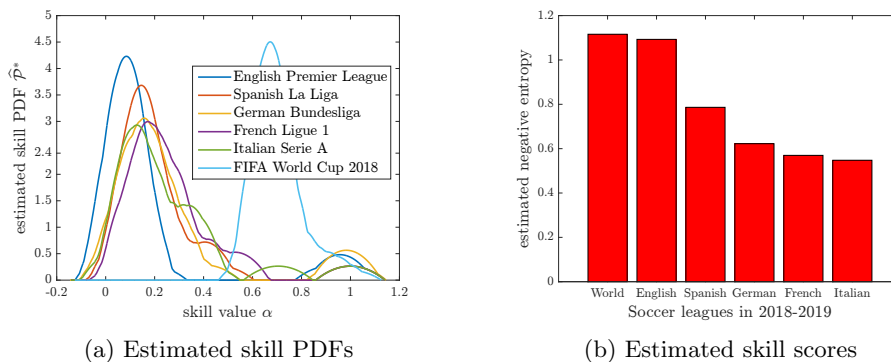


**Figure 2** Plots 2a–2b and 2c illustrate the *estimated skill PDFs* and corresponding *estimated negative differential entropies*, respectively, of FIFA Soccer World Cups from 1970 to 2018.

## 6.5. European Soccer Leagues

Yet again, we use publicly available data from Wikipedia for the English Premier League (EPL), Spanish La Liga, German Bundesliga, French Ligue 1, and Italian Serie A in the 2018-2019 season. Each league has between  $n = 18$  to  $n = 20$  teams, with every pair of teams playing 0, 1, or 2 times against each other (excluding ties). Figure 3a illustrates the skill PDFs of these leagues and the 2018 FIFA World Cup. Figure 3b sorts the negative entropies of the skill PDFs and recovers an intuitively sound ranking of these leagues. Indeed, many fans believe that EPL has better “quality” teams than other leagues (McIntyre 2019, Spacey 2020), and this observation is confirmed by Figure 3a. Figure 3a reveals that EPL has higher negative entropy than other leagues since its skill PDF has the tallest and narrowest peak, presumably because EPL mainly contains highly competitive teams with little variation among them. Moreover, as expected, the skill levels of World Cup teams are also tightly concentrated in a small interval, suggesting that the World Cup is also very competitive.

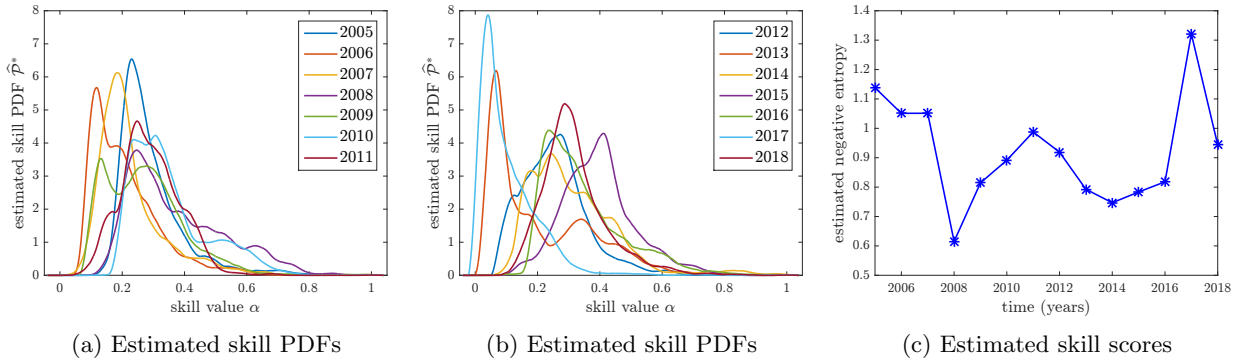
This example shows how our algorithm can be used to compare different leagues within the same sport (or even different sports).



**Figure 3** Plots 3a and 3b illustrate the *estimated skill PDFs* and corresponding *estimated negative differential entropies*, respectively, of European soccer leagues in the 2018-2019 season (as well as the FIFA World Cup in 2018).

## 6.6. US Mutual Funds

Our final experiments are calculated based on the *CRSP US Survivor-Bias-Free Mutual Funds Database* that is made available by the Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business, and can be obtained through the Wharton Research Data Services (WRDS). We consider  $n = 3260$  mutual funds in this dataset that have monthly net asset values recorded from January 2005 to December 2018. These values are preprocessed by computing monthly returns (i.e., change in net asset value normalized by the previous month's value) for all funds, which provide a fair measure of monthly performance. Then, we perceive each year as a tournament where each fund plays  $k = 12$  monthly games against every other fund, and one fund beats another in a month if it has a larger monthly return. Figures 4a and 4b depict the skill PDFs obtained by applying our algorithm to the win-loss data (produced by the method above) every year, and Figure 4c presents the associated negative entropies. Clearly, 2017 and the Great Recession in 2008 were the times where negative entropy was maximized and minimized, respectively, in Figure 4c. Figures 4a and 4b unveil that the skill PDF is much more spread out in 2008 compared to 2017, which contains a large peak near 0. So, as expected, far fewer lowly skilled funds existed during the economic recession in 2008. Furthermore, the flatter skill distributions in the few years post 2008 reveal that mutual funds became more competent due to the financial crisis. These observations elucidate the utility of our algorithm in identifying and explaining trends in other kinds of data, such as financial data.



**Figure 4** Plots 4a–4b and 4c illustrate the *estimated skill PDFs* and corresponding *estimated negative differential entropies*, respectively, of US mutual funds from 2005 to 2018.

## 7. Conclusion

In this paper, we presented and analyzed a statistical model that enabled us to rigorously quantify the overall “quality” of relative skill levels in a tournament. Specifically, we assumed that the outcomes of pairwise games between agents are determined by the BTL model, and the skill parameters of this model are drawn from an unknown PDF belonging to a non-parametric class. We then proposed an efficient two-step algorithm to learn these skill distributions from win-loss data of tournaments. Furthermore, we established tight minimax bounds on the skill parameter estimation step of our method, and then proved that our entire algorithm is minimax near-optimal in the MSE sense when the skill PDF is smooth. Finally, using the negative entropy of a learnt distribution as an overall skill score, we demonstrated the utility of our algorithm in rigorously discerning and corroborating various trends in sports and financial data.

In closing, we suggest some broad future directions. Firstly, it would be useful to develop minimax optimal algorithms that directly estimate meaningful overall skill scores from tournament data, e.g., KL divergence or Wasserstein distance between  $P_\alpha$  and some fixed distribution. Secondly, the BTL model for pairwise comparisons can be too simplistic in certain real-world scenarios. In these cases, it may be suitable to analyze the estimation of skill distributions for other models of pairwise comparisons from the literature, such as the (generalized) Thurstonian model (cf. [Thurstone 1927](#)) or more general stochastically transitive models (cf. [Chatterjee 2015](#)). Thirdly, the aforementioned issue could also be partly addressed by deriving “good” hypothesis tests that verify whether observed tournament data truly obeys a BTL model.

## Appendix A: Proofs of Auxiliary Results

In this appendix, we provide proofs of several auxiliary results.



### A.1. Proof of Proposition 1

*Proof.* First, define the random variables:

$$\forall m \in [n], M_m \triangleq \sum_{r \in [n] \setminus \{m\}} \mathbb{1}\{\{m, r\} \in \mathcal{G}(n, p)\}$$

which count the numbers of outcomes of games observed for each player. Furthermore, define the event:

$$A \triangleq \{\forall m \in [n], M_m \leq 2(n-1)p\}.$$

Then, we can verify that  $A \subseteq \{S \in \mathcal{S}_{n \times n}\}$ . Indeed, if  $A$  occurs, then we have:

$$\begin{aligned} & \forall m \in [n], \frac{1}{2np} \sum_{r \in [n] \setminus \{m\}} \mathbb{1}\{\{m, r\} \in \mathcal{G}(n, p)\} \leq \frac{M_m}{2(n-1)p} \leq 1 \\ \Rightarrow & \forall m \in [n], \frac{1}{2np} \sum_{r \in [n] \setminus \{m\}} Z(m, r) \leq 1 \\ \Leftrightarrow & \forall m \in [n], S(m, m) \geq 0 \\ \Leftrightarrow & S \in \mathcal{S}_{n \times n} \end{aligned}$$

where the second line follows from (3), and the third and fourth lines follow from (6). Hence, it suffices to prove that  $\mathbb{P}(A^c) \leq n^{-c_1}$ . To this end, notice that:

$$\begin{aligned} \mathbb{P}(A^c) &= \mathbb{P}(\exists m \in [n], M_m > 2(n-1)p) \\ &\leq n \mathbb{P}\left(\frac{1}{n-1} \sum_{r=2}^n \mathbb{1}\{\{1, r\} \in \mathcal{G}(n, p)\} - p \geq p\right) \\ &\leq n \exp\left(-\frac{3(n-1)p}{8}\right) \\ &\leq n \exp\left(-\frac{2(c_1+1)(n-1)\log(n)}{n}\right) \\ &\leq \frac{1}{n^{c_1}} \end{aligned}$$

where the second inequality follows from the union bound and the fact that  $\{M_m : m \in [n]\}$  are identically distributed random variables, the third inequality follows from Lemma 11 in Appendix B since each indicator random variable  $\mathbb{1}\{\{1, r\} \in \mathcal{G}(n, p)\}$  has mean  $p$ , variance  $p(1-p) \leq p$ , and satisfies the bound  $|\mathbb{1}\{\{1, r\} \in \mathcal{G}(n, p)\} - p| \leq 1$ , the fourth inequality follows from our assumption that  $p \geq 16(c_1+1)\log(n)/(3n)$ , and the fifth inequality holds because  $n \geq 2$ . This completes the proof.  $\square$

## A.2. Proof of Proposition 4

*Proof.* This is a corollary of Theorem 3. Observe that for any  $P_\alpha \in \mathcal{P}$  and any (Borel measurable) function  $f: \mathbb{R} \rightarrow [-1, 1]$  bounded by 1, we have:

$$\begin{aligned} \mathbb{E} \left[ \left( \int_{\mathbb{R}} f(x) \widehat{\mathcal{P}}^*(x) dx - \int_{\mathbb{R}} f(x) P_\alpha(x) dx \right)^2 \right] &= \mathbb{E} \left[ \left( \int_{\mathbb{R}} f(x) (\widehat{\mathcal{P}}^*(x) - P_\alpha(x)) dx \right)^2 \right] \\ &\leq \mathbb{E} \left[ \left( \int_{\mathbb{R}} |\widehat{\mathcal{P}}^*(x) - P_\alpha(x)| dx \right)^2 \right] \\ &= \mathbb{E} \left[ \left( \int_{-1}^2 |\widehat{\mathcal{P}}^*(x) - P_\alpha(x)| dx \right)^2 \right] \\ &\leq 3 \mathbb{E} \left[ \int_{-1}^2 (\widehat{\mathcal{P}}^*(x) - P_\alpha(x))^2 dx \right] \end{aligned}$$

where the second inequality follows from Hölder's inequality, the third equality holds because the bandwidth parameter in (10) satisfies  $h \in (0, 1]$  for all sufficiently large  $n \in \mathbb{N}$  (and hence,  $\widehat{\mathcal{P}}^*$  has support in  $[-1, 2]$ ), and the fourth inequality follows from the Cauchy-Schwarz-Bunyakovsky inequality. Then, taking suprema over  $P_\alpha \in \mathcal{P}$  and  $f: \mathbb{R} \rightarrow [-1, 1]$  on both sides yields:

$$\begin{aligned} \sup_{P_\alpha \in \mathcal{P}} \sup_{f: \mathbb{R} \rightarrow [-1, 1]} \mathbb{E} \left[ \left( \int_{\mathbb{R}} f(x) \widehat{\mathcal{P}}^*(x) dx - \int_{\mathbb{R}} f(x) P_\alpha(x) dx \right)^2 \right] &\leq 3 \sup_{P_\alpha \in \mathcal{P}} \mathbb{E} \left[ \int_{\mathbb{R}} (\widehat{\mathcal{P}}^*(x) - P_\alpha(x))^2 dx \right] \\ &\leq 3 c_{12} \max \left\{ \left( \frac{1}{\delta^2 p k} \right)^{\frac{\eta}{\eta+1}}, 1 \right\} \left( \frac{\log(n)}{n} \right)^{\frac{\eta}{\eta+1}} \end{aligned}$$

where the second inequality follows from Theorem 3. This completes the proof.  $\square$

## A.3. Proof of Claim 5

*Proof.* This is a straightforward exercise in calculus. Indeed, we have for all  $x, y \geq \delta$ :

$$\left| \frac{\partial F}{\partial y}(x, y) \right| = \frac{x}{(x+y)^2} \quad \text{and} \quad \left| \frac{\partial F}{\partial x}(x, y) \right| = \frac{y}{(x+y)^2},$$

which implies that:

$$\max_{x, y \geq \delta} \left| \frac{\partial F}{\partial y}(x, y) \right| = \max_{x, y \geq \delta} \left| \frac{\partial F}{\partial x}(x, y) \right| = \max_{t \geq \delta} \frac{t}{(t+\delta)^2} = \frac{1}{4\delta},$$

where the final equality holds because it is easy to verify that the map  $\delta \leq t \mapsto t/(t+\delta)^2$  is globally maximized at  $t = \delta$ . This establishes the Lipschitz constants in parts 1 and 2 of Claim 5.  $\square$

## A.4. Proof of Lemma 8

*Proof.* As before, starting with (52) and the fact that  $\|\pi\|_1 = 1$ , observe that:

$$\forall t > 0, \mathcal{L}_1(t) = \sup_{\nu = (\nu_1, \dots, \nu_n) \in \mathcal{S}_n} \mathbb{P}(\|\pi - \nu\|_1 \leq t)$$

$$\begin{aligned}
&\leq \sup_{\nu=(\nu_1, \dots, \nu_n) \in \mathcal{S}_n} \mathbb{P}(\|\tilde{\pi} - (\nu_1, \dots, \nu_{n-1})\|_1 \leq t) \\
&= \sup_{\nu=(\nu_1, \dots, \nu_n) \in \mathcal{S}_n} \int_{\mathbb{R}^{n-1}} P_{\tilde{\pi}}(\tau) \mathbb{1}\{\|\tau - (\nu_1, \dots, \nu_{n-1})\|_1 \leq t\} d\tau \\
&\leq \frac{n^{n-1}}{(1-\delta)^n} \sup_{\nu=(\nu_1, \dots, \nu_n) \in \mathcal{S}_n} \underbrace{\int_{\mathbb{R}^{n-1}} \mathbb{1}\{\|\tau - (\nu_1, \dots, \nu_{n-1})\|_1 \leq t\} d\tau}_{\text{volume of } \ell^1\text{-ball with radius } t} \\
&= \frac{n^{n-1} 2^{n-1}}{(1-\delta)^n (n-1)!} t^{n-1} \\
&\leq \frac{1}{5\sqrt{n}} \left(\frac{2e}{1-\delta}\right)^n t^{n-1}
\end{aligned}$$

where the second inequality uses (53) and the fact that  $\|\tilde{\pi} - (\nu_1, \dots, \nu_{n-1})\|_1 \leq \|\pi - \nu\|_1$  (and this bound is not too loose because  $\|\pi - \nu\|_1 \leq 2\|\tilde{\pi} - (\nu_1, \dots, \nu_{n-1})\|_1$  via the triangle inequality), the fourth inequality follows from Lemma 6, the fifth equality uses the well-known volume of the  $\ell^1$ -ball (or cross-polytope) with radius  $t$ , and the sixth inequality follows from the Stirling's formula bound (see, e.g., Feller 1968, Chapter II, Section 9, Equation (9.15)):

$$n! \geq \frac{5}{2} \sqrt{n} \left(\frac{n}{e}\right)^n.$$

This completes the proof.  $\square$

## Appendix B: Concentration of Measure Inequalities

In this appendix, we present two well-known exponential concentration of measure inequalities that are used in this paper. The first of these results bounds the tail probability of the empirical average of a collection of independent bounded random variables.

**LEMMA 10 (Hoeffding's Inequality (Hoeffding 1963, Theorems 1 and 2)).** *Given independent random variables  $X_1, \dots, X_n \in [a, b]$ , for some constants  $a < b$ , we have for every  $\varepsilon \geq 0$ :*

$$\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[X_i] \geq \varepsilon\right) \leq \exp\left(-\frac{2n\varepsilon^2}{(b-a)^2}\right).$$

The second of these results provides a tighter bound on the tail probability of the empirical average of a collection of independent bounded random variables using information about the variances of the random variables.

**LEMMA 11 (Bernstein's Inequality (Bernstein 1946)).** *Given independent random variables  $X_1, \dots, X_n$  such that for some constants  $a, b > 0$ ,  $|X_i - \mathbb{E}[X_i]| \leq a$  and  $\text{VAR}(X_i) \leq b$  for all  $i \in [n]$ , we have for every  $\varepsilon \geq 0$ :*

$$\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[X_i] \geq \varepsilon\right) \leq \exp\left(-\frac{n\varepsilon^2}{2b + \frac{2}{3}a\varepsilon}\right).$$

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